Assignment 4, CAAM/STAT 581, due October 8

1. Let $\Omega = \Omega' = [0, \infty)$ and let $\mathcal{B} = \mathcal{B}' = \text{Borel}$ sets.

   a. For each of the following functions, find the $\sigma$-field it generates.

      (i) $h(x) = x$

      (ii) $h(x) = x^2$

      (iii) $h(x) = 1$ if $x$ is rational, $h(x) = 0$ otherwise

   b. Give an example of a function $h$ which is not measurable.

2. Let $\Omega = [0, \infty)$ and let $\mathcal{C}$ be the class of singletons, $\mathcal{C} = \{\{x\}, x \geq 0\}$. Let $\mathcal{A} = \{B \subseteq \mathbb{R} : B \text{ is countable or } B^c \text{ is countable}\}$, the so called countable/co-countable $\sigma$-field (see book, p.13).

   a. Show that $\mathcal{A} = \sigma(\mathcal{C})$.

   b. Give an example of a Borel set which is not in $\mathcal{A}$.

   c. Let $\Omega' = [0, \infty)$ and let $\mathcal{B}'$ be the Borel $\sigma$-field. Which of the following functions are measurable $\mathcal{A}/\mathcal{B}'$?

      (i) $h(x) = x$

      (ii) $h(x) = x$ if $x$ is rational, $h(x) = 0$ otherwise

      (iii) $h(x) = 0$ if $x$ is rational, $h(x) = x$ otherwise

      (iv) $h(x) = 1$ if $x \leq 1$, $h(x) = 0$ otherwise