Assignment 7, CAAM/STAT 581, due November

1. For the following functions, determine if the integral $\int_{-\infty}^{\infty} f(x)\,dx$ exists and if it does, compute its value.
   
   a. \[ f(x) = \begin{cases} -1/x^2 & \text{if } x \leq -1 \\ x & \text{if } -1 \leq x \leq 1 \\ 1/x^2 & \text{if } x \geq 1 \end{cases} \]

   b. \[ f(x) = \begin{cases} -1/x^2 & \text{if } x \leq -1 \\ x & \text{if } -1 \leq x \leq 1 \\ 1/x & \text{if } x \geq 1 \end{cases} \]

   c. \[ f(x) = \begin{cases} -1/x & \text{if } x \leq -1 \\ x & \text{if } -1 \leq x \leq 1 \\ 1/x & \text{if } x \geq 1 \end{cases} \]

2. Compute the limit of
   \[ n \int_{0}^{1} \frac{\sqrt{x}}{1 + n\sqrt{x}} \, dx. \]
as $n \to \infty$.

3. Compute the limit $\lim_{n \to \infty} \int_{0}^{\infty} f_n(x)\,dx$ for the following functions.

   a. \[ f_n(x) = \frac{n \sin(\frac{\pi}{x})}{x(1 + x^2)} \]

   b. \[ f_n(x) = \begin{cases} 1/n & \text{if } x \geq n \\ 0 & \text{if } x < n \end{cases} \]
4. If \( X \) is a discrete random variable with range \{1, 2, \ldots\}, then it is a well known result that
\[
E[X] = \sum_{n=1}^{\infty} P(X \geq n).
\]
Show that this result follows from Corollary 5.3.1. (write \( X \) as a sum of indicators).

5. Let \( X_1, X_2, \ldots \) be random variables such that \( X_n \geq 0 \) and \( E[X_n] \leq K < \infty \) for all \( n \) and \( X_n \to X \) a.s. Is it then true that \( E[X] \leq K \)? What if the condition \( X_n \geq 0 \) is replaced by \( E[X_n] \geq 0 \)? Proofs or counterexamples.

6. Let \((\Omega, \mathcal{B}, P)\) be a probability space, let \( C \in \mathcal{B} \) and let \( X \) be a random variable which is independent of \( C \) (meaning that \( P(X^{-1}(B) \cap C) = P(X^{-1}(B))P(C) \) for all Borel sets \( B \)). Show that \( E[X;C] = E[X]P(C) \) (start with...)

2