Final Exam, STAT 581, Fall 2003

1. Let $\mathcal{B}$ be the Borel sets on $\mathbb{R}$ and define the class $\mathcal{F} = \{ A \in \mathcal{B} : A$ is finite or $A^c$ is finite \}$ (where ”finite” as usual means having a finite number of points). Let $\mu : \mathcal{F} \to \mathbb{R}$ be a function such that $\mu(A) = 0$ if $A$ is finite and $\mu(A) = 2$ if $A^c$ is finite.

a. Show that $\mathcal{F}$ is a field but not a $\sigma$-field. What is the $\sigma$-field generated by $\mathcal{F}$?

b. Show that $\mu$ is a measure on $(\mathbb{R}, \mathcal{F})$.

2a. State the first and second Borel-Cantelli lemmas.

b. Use Borel-Cantelli to construct examples of sequences of independent random variables $X_1, X_2, ...$ such that

(i) $X_n \to 0$ a.s. and $E[X_n] \equiv 1$.

(ii) $X_n$ does not converge a.s., $E[X_n] < 1$ for all $n$ and $E[X_n] \to 1$.

(iii) $X_n \to \infty$ a.s. and $E[X_n] \equiv 0$.

3. Find $\lim_{n \to \infty} \int_1^\infty f_n(x) dx$ for the following functions (note the limits of the integral):

(i) $f_n(x) = I_{[n,n+1]}(x)$

(ii) $f_n(x) = \sin(nx)e^{-nx}x^{-2}$

(iii) $f_n(x) = xn^{-1}$

(iv) $f_n(x) = (-1)^nxn^{-2}$

(v) $f_n(x) = (1 + nx^2)(1 + x^2)^{-n}$
4a. State Fatou’s lemma for expectations.

b. Fatou’s lemma for probabilities states that if $A_1, A_2, \ldots$ is a sequence of events, then $P(\lim \inf_n A_n) \leq \lim \inf_n P(A_n)$. Show that this follows from Fatou’s lemma for expectations.

5. Let $(\Omega, \mathcal{B}, \mu)$ be a measure space. Fix a set $B \in \mathcal{B}$ and define $\nu(A) = \mu(A \cap B)$ for $A \in \mathcal{B}$.

a. Show that $\nu$ is a measure on $(\Omega, \mathcal{B})$.

b. Let $g : \Omega \to \mathbb{R}$ be a function such that $\int_B gd\mu$ exists. Show that

$$\int_B gd\mu = \int_\Omega g d\nu.$$

6a. Let $X$ be a non-negative random variable. Show that

$$E[X] = \int_0^\infty (1 - F(x))dx.$$

*Hint:* Recall that if $X$ has range $\{1, 2, \ldots\}$, then $E[X] = \sum_n P(X \geq n)$ which can be viewed as changing the order of integration. The same type of method applies here.

b. Let $X_1, X_2, \ldots$ be i.i.d. non-negative random variables with infinite expectation. Show that $P(X_n > n \text{ i.o.}) = 1$ (you may use the result in a even if you did not manage to prove it).