Assignment 5, STAT 582, due April 21

1. Let $X_1, X_2, \ldots$ be i.i.d. random variables such that

$$X_n = \begin{cases} 
0 & \text{with probability } 1/2 \\
2 & \text{with probability } 1/2, 
\end{cases}$$

let $Y_n = X_1 \cdot X_2 \cdot \ldots \cdot X_n$ and $\mathcal{B}_n = \sigma(X_1, \ldots, X_n)$.

a. Show that $Y_n$ is a martingale with respect to $\mathcal{B}_n$.

b. Does $Y_n$ converge almost surely? If so, to what? Does $Y_n$ converge in $L_1$?

2. Let $S_n$ be a symmetric simple random walk starting in 0 and let $\nu = \inf \{ n : S_n = -a \text{ or } S_n = a \}$ (symmetric absorbing barriers). Let $Y_n = S_n^4 - 6nS_n^2 + 3n^2 + 2n$.

a. Show that $Y_n$ is a martingale (with respect to $\mathcal{B}_n = \sigma(X_0, \ldots, X_n)$).

b. It can be shown that the Optional Stopping Theorem applies to $Y_n$. Use this to compute $Var[\nu]$.

3. Let $\nu$ and $\tau$ be stopping times. Which of the following are stopping times?

a. $\min(\nu, \tau)$  

b. $\max(\nu, \tau)$  

c. $\nu + \tau$  

d. $\nu + 1$  

e. $\nu - 1$

4. Let $\{Z_n\}$ be a branching process and let $q$ be the extinction probability. Let $Y_n = q^{Z_n}$ and let $\mathcal{B}_n = \sigma(Z_0, \ldots, Z_n)$. Show that $Y_n$ is a martingale w.r.t. $\mathcal{B}_n$.

b. Does $Y_n$ converge almost surely? To what? Does $Y_n$ converge in $L_1$?