Solutions to Assignment 6, STAT 582

1a. Since $Y_n$ is a function of $X_1, ..., X_n$ it is measurable with respect to $\mathcal{B}_n$. Further,

$$E[Y_{n+1}|\mathcal{B}_n] = E[Y_n X_{n+1}|\mathcal{B}_n] = Y_n E[X_{n+1}] = Y_n$$

since $Y_n$ is measurable with respect to $\mathcal{B}_n$, $X_{n+1}$ is independent of $\mathcal{B}_n$ and $E[X_{n+1}] = 1$.

b. Since the event $\{X_n = 0$ for some $n\}$ has probability one, $Y_n \to 0$ a.s. Since $E[|Y_n - 0|] = E[Y_n] = 1$, $Y_n$ does not converge in $L_1$.

2a. Use the facts that $S_{n+1} = S_n + X_{n+1}$ and that the $X_k$ are i.i.d. with mean 0 to show that

$$E[S_{n+1}^4|\mathcal{B}_n] = S_n^4 + 6S_n^2 + 1$$

and

$$E[S_{n+1}^2|\mathcal{B}_n] = S_n^2 + 1.$$

Hence,

$$E[Y_{n+1}|\mathcal{B}_n] = S_n^4 + 6S_n^2 + 1 - 6(n + 1)(S_n^2 + 1) + 3(n + 1)^2 + 2(n + 1)$$

$$= S_n^4 - 6nS_n^2 + 3n^2 + 2n = Y_n.$$

b. By OST, $E[Y_n] = E[Y_0] = 0$ i.e.

$$0 = E[S_{n}^4] - 6E[\nu S_n^2] + 3E[\nu^2] + 2E[\nu] = a^4 - 6a^2 a^2 + 3E[\nu^2] + 2a^2$$

where we have used the facts that $E[\nu] = a^2$, $S_n^2 \equiv a^2$, $S_n^4 \equiv a^4$. This gives

$$E[\nu^2] = \frac{1}{3}(5a^4 - 2a^2)$$
which gives
\[
Var[\nu] = \frac{1}{3}(5a^4 - 2a^2) - a^4 = \frac{2}{3}(a^4 - a^2).
\]

3. All except e are stopping times. First note that \( \mathcal{B}_n \) is an increasing sequence and therefore \( \{ \nu = k \} \in \mathcal{B}_k \subset \mathcal{B}_n, k = 1, ..., n \) (similarly for \( \tau \)).

a. \( \{ \min(\sigma, \tau) = n \} = (\{ \sigma = n \} \cap \{ \tau \geq n \}) \cup (\{ \sigma \geq n \} \cap \{ \tau = n \}) \in \mathcal{B}_n \)

b. As in a. with \( \cup \) instead of \( \cap \) and vice versa.

c. \( \{ \nu + \tau = n \} = \sum_{k=0}^{n} \{ \nu = k \} \cap \{ \tau = n - k \} \in \mathcal{B}_n \).

d. \( \{ \tau + 1 = n \} = \{ \tau = n - 1 \} \in \mathcal{B}_n \).

e. \( \{ \tau - 1 = n \} = \{ \tau = n + 1 \} \), not necessarily in \( \mathcal{B}_n \).

4a. Since \( Z_{n+1} = \sum_{k=1}^{Z_n} X_k \) where the \( X_k \) are i.i.d we get
\[
E[Y_{n+1}|\mathcal{B}_n] = E[q^{Z_{n+1}}|Z_n] = \prod_{k=1}^{Z_n} E[q^{X_k}] = q^{Z_n} = Y_n
\]
where we use the fact that \( E[q^{X_k}] = \phi(q) = q \).

b. Yes. Recall that \( Z_n \to Z \) a.s. where \( Z = 0 \) w.p. \( q \) and \( Z = \infty \) w.p. \( 1 - q \). Hence \( Y_n \to Y \) a.s where \( Y = 1 \) w.p. \( q \) and \( Y = 0 \) w.p. \( 1 - q \). Note that the limit here is a random variable, not a constant unless \( q = 1 \), in which case \( Y_n \equiv 1 \) and \( q = 0 \), in which case \( Y_n \equiv 0 \).

Since \( |Y_n - Y| \to 0 \) and \( |Y_n - Y| \leq |Y_n| + |Y| \leq 2 \), DCT applies to give \( E[|Y_n - Y|] \to 0 \) i.e. \( Y_n \overset{L_1}{\to} Y \).

The following argument that \( Y_n \) does not converge in \( L_1 \) is INCORRECT: Since \( Y = 1 \) with probability \( q \) and 0 with probability \( 1 - q \), we get
\[ E[|Y_n - Y|] = E[|Y_n - 1|]q + E[|Y_n - 0|](1 - q) = \\
(1 - E[Y_n])q + E[Y_n](1 - q) = 2q(1 - q) \neq 0. \]

The problem is that \( Y \) and \( Y_n \) are DEPENDENT so that when conditioning on \( Y \), the mean \( E[|Y_n - Y|] \) changes. Thus, \( E[|Y_n - Y||Y = 0] \) is NOT equal to \( E[|Y_n|] = q \); conditioning on \( Y = 0 \) means that we condition on \( Z_n \rightarrow \infty \) which changes the distribution of \( Z_n \) (it can for example certainly not be 0 anymore) and hence that of \( Y_n \). Think about this!