ELEC 533 Homework 3

Due date: In class on Friday, September 28th, 2001

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11. We return to the simplified Roulette (without “zero”): \( \Omega = \{1, \ldots, 36\} \), \( P([n]) = 1/36 \) for all \( n \in \Omega \). There are 3 events we are interested in: \( E \) are the even numbers, \( R \) are the red numbers, and \( F \) are the numbers in the first row, i.e. \( F = \{1, 4, 7, 10, 13, 16, 19, 22, 25, 28, 31, 34\} \). Unlike true roulette let us assume that the red numbers are \( R = \{1, 2, 3, 7, 8, 12, 13, 14, 15, 19, 20, 24, 25, 26, 27, 31, 32, 36\} \):

\[
\begin{array}{cccccccccccc}
1 & 4 & 7 & 10 & 13 & 16 & 19 & 22 & 25 & 28 & 31 & 34 \\
2 & 5 & 8 & 11 & 14 & 17 & 20 & 23 & 26 & 29 & 32 & 35 \\
3 & 6 & 9 & 12 & 15 & 18 & 21 & 24 & 27 & 30 & 33 & 36
\end{array}
\]

Consider the random variables \( X, Y \) and \( Z \) given by:

\[
X(\omega) = \begin{cases} 
4 & \text{if } \omega \text{ is even} \\
0 & \text{else}
\end{cases} \quad Y(\omega) = \begin{cases} 
7 & \text{if } \omega \text{ is red} \\
0 & \text{else}
\end{cases} \quad Z(\omega) = \begin{cases} 
-3 & \text{if } \omega \text{ is in the first row} \\
0 & \text{else}
\end{cases}
\]

(a) Compute the marginal distributions of \( X, Y \) and \( Z \), i.e., compute \( P[X = t], P[Y = t], P[Z = t] \) for all \( t \in \mathbb{R} \).

(b) Compute the pairwise joint marginal distributions of \( (X,Y) \), \( (X,Z) \) and \( (Y,Z) \), i.e., compute \( P[X = s \text{ and } Y = t], P[X = s \text{ and } Z = t], P[Y = s \text{ and } Z = t] \) for all \( s, t \in \mathbb{R} \).

(c) Compute the full joint marginal distributions of \( (X,Y,Z) \), i.e., compute \( P[X = s \text{ and } Y = t \text{ and } Z = u] \) for all \( s, t, u \in \mathbb{R} \).

(d) Are the random variables \( X \) and \( Y \) independent?

(e) Are the random variables \( X \) and \( Z \) independent?

(f) Are the random variables \( Y \) and \( Z \) independent?

(g) Are the random variables \( X, Y \) and \( Z \) independent?

(h) Let us assume that instead of the first row, \( Z(\omega) \) equals \(-3\) if the number \( \omega \) is in the third row, i.e. in \( \{3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36\} \). Of all the questions in 11a to 11g, exactly two have now a different answer. Which are they, and what are the new answers?

The main lesson of this problem is: Given three random variables, it is not enough to check pairwise independence to decide whether the three random variables are independent. Also, the joint distribution \( F_{XYZ} \) of three variables can not be computed from the pairwise joint distributions \( F_{XY}, F_{XZ} \) and \( F_{YZ} \) since different joint distribution functions \( F_{XYZ} \) can have the same pairwise joint marginals \( F_{XY}, F_{XZ} \) and \( F_{YZ} \).

12. Compute expectation and variance of the following random variables:

(a) \( X \approx \mathcal{U}(0, 2\pi) \): Uniform on \([0, 2\pi]\)

\[
f_X(x) = \begin{cases} 
\frac{1}{2\pi} & \text{for } x \in [0, 2\pi], \\
0 & \text{otherwise.}
\end{cases}
\]

(b) \( X \approx \text{Cauchy} \):

\[
f_X(x) = \frac{1}{\pi(1 + x^2)} \quad \text{for every } x \in \mathbb{R}.
\]

(c) \( N \approx \text{Pois}(\lambda) \): Poisson with parameter \( \lambda > 0 \), which is given by

\[
P[N = n] = P_N(n) = \begin{cases} 
e^{-\lambda} \frac{\lambda^n}{n!} & \text{for } n = 0, 1, 2, \cdots \\
0 & \text{otherwise.}
\end{cases}
\]
(d) $X \sim \mathcal{N}(\mu, \sigma^2)$: Gaussian or normal distribution with parameters $\sigma^2 > 0$ and $\mu \in \mathbb{R}$, which is given through the density

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

(You don’t have to show that this is indeed a probability density.)

(e) $X \sim \exp(\lambda)$: One sided exponential with parameter $\lambda > 0$, which is given through the density

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

13. Assume that $X$ and $Y$ are independent Gaussian random variables with zero mean and variance 1.

Compute the distribution of the random variable $Z = \exp\left(-\frac{X^2 + Y^2}{2}\right)$.

Hint: the transformation from Cartesian to polar coordinates goes as $x = r \sin(\phi)$, $y = r \cos(\phi)$, $dx\,dy = rd\rho\,d\phi$.

14. Given is a r.v. $X$ which is uniformly distributed on $[0, \pi]$. We are interested in the r.v. $Y = g(X)$ where $g(t) = \sin(t)$.

(a) Compute $\mathbb{E}[\sin(X)] = \int \sin(x) f_X(x) \, dx$.

(b) Compute the pdf (density) $f_Y$ of $Y$. (You might find it convenient to compute first the CDF $F_Y$ of $Y$.)

(c) Compute $\mathbb{E}[Y] = \int y f_Y(y) \, dy$. Check whether you got the same answer as in (a).

15. (a) Let $X$ be a continuous r.v. Using the definition of expectation and the rules of integration derive

$$\mathbb{E}[aX + b] = a \mathbb{E}[X] + b,$$

where $a$ and $b$ are constants. Also show that

$$\mathbb{E}[(aX + b)^2] = a^2 \mathbb{E}[X^2] + 2ab \mathbb{E}[X] + b^2.$$

Conclude that $\text{var}(aX + b) = a^2 \text{var}(X)$.

(b) Suppose $X \sim \mathcal{N}(\mu, \sigma^2)$ and $Y = aX + b$. The mean and variance of $Y$ can be computed using 15a. Find the p.d.f of $Y$.

(c) Repeat 15b for $X \sim U(0, 2\pi)$.