32. Let \( \{N_t, t \in \mathbb{R}\} \) be a Poisson process with intensity \( \lambda \).

(a) Compute the second order joint statistics \( F_{N_t, N_s} \), i.e. compute \( P[N_t = n, N_s = m] \).

Hint: Increments.

33. Suppose \( X_i (i \in \mathbb{N}) \) is a sequence of iid random variables and suppose

\[
P[X_i = 1] = P[X_i = -1] = 1/2
\]

for all \( i \). Define a random process \( \{Y_t, t \in \mathbb{R}\} \) by

\[
Y_t = X_i \quad \text{for all } t \text{ such that } i - 1 < t \leq i.
\]

(a) Sketch a typical sample path of \( \{Y_t, t \in \mathbb{R}\} \).

(b) Find the mean and autocorrelation function of \( \{Y_t, t \in \mathbb{R}\} \). Is the process wide sense stationary (wss)? Show your argument.

(c) Suppose \( A \) is a r.v. independent of the \( X_i \) and uniformly distributed in \([0, 1]: A \sim U([0, 1])\). Define the random process \( \{Y_t, t \in \mathbb{R}\} \) by

\[
Z_t = Y_t + A
\]

Sketch a typical sample path of \( \{Z_t, t \in \mathbb{R}\} \).

(d) Is \( Z_t \) wss? Is \( Z_t \) second order stationary? Why or why not?

34. Compute the spectral density for processes with the following auto-correlation:

(a) \( R_X(\tau) = \exp(-\beta|\tau|) \).

(b) \( R_X(\tau) = \max(1 - |\tau|/T, 0) \). HINT: \( R_X \) is the convolution of two extremely simple functions with extremely simple Fourier transforms.

The first example is the auto-correlation of a Gauss-Markov process, the second can be obtained running white noise through an integral averager (see class).

35. Let \( \{X_t\} \) be a continuous time, but discrete valued Markov process. For simplicity we assume that the values are integers. As in the case with continuous values treated in class, we define the process via its initial distribution \( P[X_0 = k] \) at time 0 and the transition probabilities

\[
P[X_t = k|X_s = n]
\]

for any \( t > s \). More precisely, the marginals are

\[
P[X_t = k] = \sum_{n=1}^{\infty} P[X_t = k|X_0 = n] \cdot P[X_0 = n]
\]

and the f.d.d. are

\[
P[X_{t_1} = k_1, \ldots X_{t_n} = k_n] = P[X_{t_n} = k_n|X_{t_{n-1}} = k_{n-1}] \cdots P[X_{t_2} = k_2|X_{t_1} = k_1] \cdot P[X_{t_1} = k_1]
\]

As in the continuous-valued case done in class, this defines a consistent f.d.d. if and only if the Chapman-Kolmogorov equations are satisfied. Here, these Chapman-Kolmogorov equations read as:

\[
\sum_{n=1}^{\infty} P[X_t = k|X_s = n] \cdot P[X_s = n|X_r = m] = P[X_t = k|X_r = m].
\]
(a) Check that the Chapman-Kolmogorov equations imply consistency. To this end, assume that the Chapman-Kolmogorov equations hold and show that for any \( j = 1, \ldots, n \)

\[
\sum_{k_j=1}^{\infty} P[X_{t_j} = k_1, \ldots, X_{t_n} = k_n] = P[X_{t_1} = k_1, \ldots, X_{t_{j-1}} = k_{j-1}, X_{t_{j+1}} = k_{j+1}, \ldots, X_{t_n} = k_n].
\]

(b) Consider the following particular transition probability where \( t > s \) and where \( k \) and \( n \) are positive integers:

\[
P[X_t = k | X_s = n] = \begin{cases} 
0 & \text{if } k < n \\
\frac{e^{-\lambda(t-s)}(\lambda(t-s))^{k-n}/(k-n)!}{\lambda^k} & \text{if } k \geq n.
\end{cases}
\]

In other words, given that \( X_s = n \) the increment \( X_t - X_s \) is a Poisson random variable of mean \( \lambda(t-s) \).

Verify that the Chapman-Kolmogorov equations are satisfied. Hint 1: recall that the Chapman-Kolmogorov equations hold and show that for any \( n \)

\[
\sum_{k=0}^{\infty} P[X_{t_1} = k | X_0 = 0] = 1.
\]

Hint 2: be careful about noting when \( P[X_t = k | X_s = n] = 0 \).

(c) With the same transition probabilities as before add the initial distribution \( P[X_0 = 0] = 1 \).

Show that \( X_t \) is a Poisson variable of mean \( \lambda t \).

Note: In this case, it is possible to show that \( X_t \) is actually a Poisson renewal process. This is not obvious, since one needs to construct arrival times \( T_k \) of events which should be spaced by independent waiting times. A first step towards this goal would be to show that the increments \( X_{t+k} - X_k \) over disjoint time intervals are independent.

**Here are some more problems to practice your skill. They are not mandatory.**

36. Suppose \( A \) and \( B \) are independent Gaussian random variables, i.e. \( A \sim \mathcal{N}(0, 1) \) and \( B \sim \mathcal{N}(0, 1) \). Define the random process \( \{X_t, t \in \mathbb{R}\} \) by

\[
X_t = A + Bt + t^2.
\]

(a) Compute the following marginal distributions: \( \phi_{X_t} \) for arbitrary \( t \) and \( \phi_{X_1, X_5} \).

(b) Are \( X_1, X_2, \ldots, X_5 \) jointly Gaussian?

(c) Find \( \mathbb{E}[X_5 | X_0] \). This is a prediction of \( X_5 \) knowing \( X_0 \), and it is a random variable. Compute its variance.

(d) Find \( \mathbb{E}[X_5 | X_0, X_1] \). This is a prediction of \( X_5 \) knowing \( X_0 \) and \( X_1 \), and it is also a random variable. Compute its variance.

37. Throughout let all processes be wss. Recall: for \( X_t \) the power spectral density \( S_X(\nu) \) is defined as the Fourier transform of the auto-correlation function \( R_X(\tau) = \mathbb{E}[X_{t+\tau}X_t^*] \).

(a) Let \( U_t \) and \( V_t \) be orthogonal wss processes, meaning that \( \mathbb{E}[U_tV_s] = 0 \) for all \( t \) and \( s \). Set \( W_t = U_t + V_t \). Show that \( R_{W}(\tau) = R_{U}(\tau) + R_{V}(\tau) \) and conclude that \( S_W(\nu) = S_U(\nu) + S_V(\nu) \).

(b) If \( Y_t = aX_t \) then \( R_Y(\tau) = |a|^2 R_X(\tau) \) and \( S_Y(\nu) = |a|^2 S_X(\nu) \).

(c) Let \( Y_t = X_t \cdot \cos(\nu_0 t + \Theta) \) where \( \nu_0 \) is a fixed frequency and where the random phase \( \Theta \) is independent of the process \( X_t \) and uniformly distributed on \([\pi, \pi]\). Show that \( R_Y(\tau) = R_X(\tau) \cos(\nu_0 \tau) \) and \( S_Y(\nu) = \frac{1}{2} (S_X(\nu + \nu_0) + S_X(\nu - \nu_0)) \).

(d) Assume that \( X_t \) is a zero mean process, meaning that \( \mu_X(t) = 0 \). Set \( Z_t = X_t + b \). Show that \( R_Z(\tau) = R_X(\tau) + |b|^2 \) and \( S_Z(\nu) = S_X(\nu) + 2\pi|b|^2 \delta(\nu) \).

(e) Assume that \( A_t \) is a wss process such that its auto-correlation function is related to that of \( X_t \) via \( R_A(\tau) = -R''_X(\tau) \). Show that \( S_A(\nu) = \nu^2 S_X(\nu) \). (Here, \( R'' \) denotes the second derivative.)

Most of these questions can be derived from properties of the Fourier transform.