14. Assume that $X$ and $Y$ are independent Gaussian random variables with zero mean and variance 1. Compute the distribution of the random variable $Z = \exp\left(-\frac{X^2 + Y^2}{2}\right)$. Hint: the transformation from Cartesian to polar coordinates goes as $x = r \sin(\phi)$, $y = r \cos(\phi)$, $dx\,dy = rdr\phi$.

15. Suppose $X$ and $Y$ are joint r.v.’s with

$$f_{XY}(x, y) = \begin{cases} e^{-y} & x > 0, y > x \\ 0 & \text{otherwise} \end{cases}$$

(a) Show that $\iint f_{XY}(x, y) \, dy\,dx = 1$.
(b) Find $f_X$ and $f_Y$.
(c) Are $X$ and $Y$ independent? Show your reasoning.

16. Let the joint density of the pair of random variables $(X, Y)$ be given by

$$f_{XY}(x, y) = \begin{cases} y \exp(-xy) & \text{if } x > 1 \text{ and } y > 0 \\ 0 & \text{else.} \end{cases}$$

(a) Compute the marginal densities $f_X$ and $f_Y$.
(b) Are $X$ and $Y$ independent? Show your reasoning.
(c) Compute $\mathbb{E}[X]$ and $\mathbb{E}[Y]$.

17. Two pairs of discrete random variables $(U, V)$ and $(X, Y)$ are given via their joint distributions:

$$P[U = u, V = v] = \begin{cases} 1/2 & \text{if } u = 1, v = 0 \\ 1/6 & \text{if } u = 1, v = 1 \\ 1/12 & \text{if } u = -1, v = 1 \\ 1/4 & \text{if } u = -1, v = 0 \end{cases}$$

and

$$P[X = x, Y = y] = \begin{cases} 7/12 & \text{if } x = 1, y = 0 \\ 1/12 & \text{if } x = 1, y = 1 \\ 1/6 & \text{if } x = -1, y = 1 \\ 1/6 & \text{if } x = -1, y = 0 \end{cases}$$

(a) Show that the marginals are the same, that is $F_X = F_U$ and $F_Y = F_V$.
(b) Which pair is independent?

18. Let $X$ and $Y$ be standard jointly Gaussian r.v.’s ($\mu_X = \mu_Y = 0$, $\sigma_X = \sigma_Y = 1$) with joint density

$$f_{XY}(x, y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(\frac{-x^2 - 2\rho xy + y^2}{2(1-\rho^2)}\right),$$

where $\rho$ is a constant.

(a) Show by direct computation that $\text{cov}(X, Y) / \sqrt{\text{Var}(X)\text{Var}(Y)} = \rho$.
(b) Compute the density $f_Z$ of the sum $Z = X + Y$ using the general formula for the density of the sum of (dependent) random variables: $f_Z(z) = \int f_{XY}(x, z-x)\,dx$. Conclude that $Z$ is as well Gaussian.