26. Let \( X_n \) be Gaussian r.v.-s with mean \( \mu_n \) and variance \( \sigma^2_n \). Under what conditions on the sequences \( \mu_n \) and \( \sigma^2_n \) does \( X_n \) converge in distribution and what is the limiting distribution. Hint: Use the fact that \( X_n \) converges in distribution if and only if their characteristic functions \( \phi_{X_n} \) converge. Check, under what conditions the limit of \( \phi_{X_n} \) exists and make sure that the limit is again a meaningful characteristic function.

27. Suppose that \( X_m \overset{D}{\rightarrow} X \) and that there is a constant \( c \) such that \( P[X = c] = 1 \). Show that
\[
X_m \overset{i.p.}{\rightarrow} X
\]
[HINT: You need to show that \( P[|X_m - X| > \varepsilon] \rightarrow 0 \) \((m \to \infty)\) for any \( \varepsilon \). Because \( X \) is essentially constant and equal to \( c \), this probability is equal to \( 1 - P[c-\varepsilon \leq X_m \leq c+\varepsilon] \) which is easily expressed in terms of the CDFs \( F_{X_m} \). Now, use that \( F_{X_m} \) converges to \( F_X \), which has a particularly simple form because \( X \) is essentially constant.]

28. Let \( X \) be a Bernoulli random variable which takes the values \( 1 \) and \( -1 \) both with probability \( \frac{1}{2} \).

(a) Compute the characteristic function of \( X \), i.e., \( \phi_X(u) = \mathbb{E}[\exp(iuX)] \). Find a simple expression in terms of a cosine function.

(b) Verify that \( \phi''(0) = -\mathbb{E}[X^2] \).

(c) Compute the characteristic function of
\[
Y_n = \frac{X_1 + \ldots + X_n}{2\sqrt{n}}
\]
using the above simple formula. Here, the random variables \( X_n \) are independent and of the same distribution as \( X \).

(d) Approximate the cosine function by its Taylor polynomial of order 2 (i.e., the quadratic polynomial that provides the best approximation) and compute the limit of the characteristic function of \( Y_n \). Conclude that \( Y_n \) converges in distribution. What is the limiting distribution?

29. Let us define the \( k \)-th cumulant \( \lambda_k \) as
\[
\lambda_k := (-i)^k \cdot \psi^{(k)}(0),
\]
where \( \psi^{(k)} \) denotes the \( k \)-th derivative of \( \psi(u) = \log \mathbb{E}[\exp(iuX)] \) (the logarithm of the characteristic function).

(a) Show that the cumulants of order 3 and higher for a Gaussian r.v. are zero.

(b) Express the cumulants (of all orders) of an exponential r.v. in terms of the parameter \( r \) of its p.d.f.
\[
f(x) = \begin{cases} r \cdot \exp(-rx) & \text{if } x > 0 \\ 0 & \text{else.} \end{cases}
\]

Remark: Beside mean and variance, other quantities are used to describe distributions: Skew is defined as \( \lambda_3/\lambda_2^{3/2} \) and Kurtosis as \( \lambda_4/\lambda_2^2 \). Note that Skew is zero for a symmetrical distribution. If non zero, the Skew is a measure of asymmetry. The Kurtosis measures the "weight" of the tails: the larger, the heavier the tails. Skew and Kurtosis for a Gaussian r.v. are both zero, thus, a Gaussian has light tails and is symmetrical.