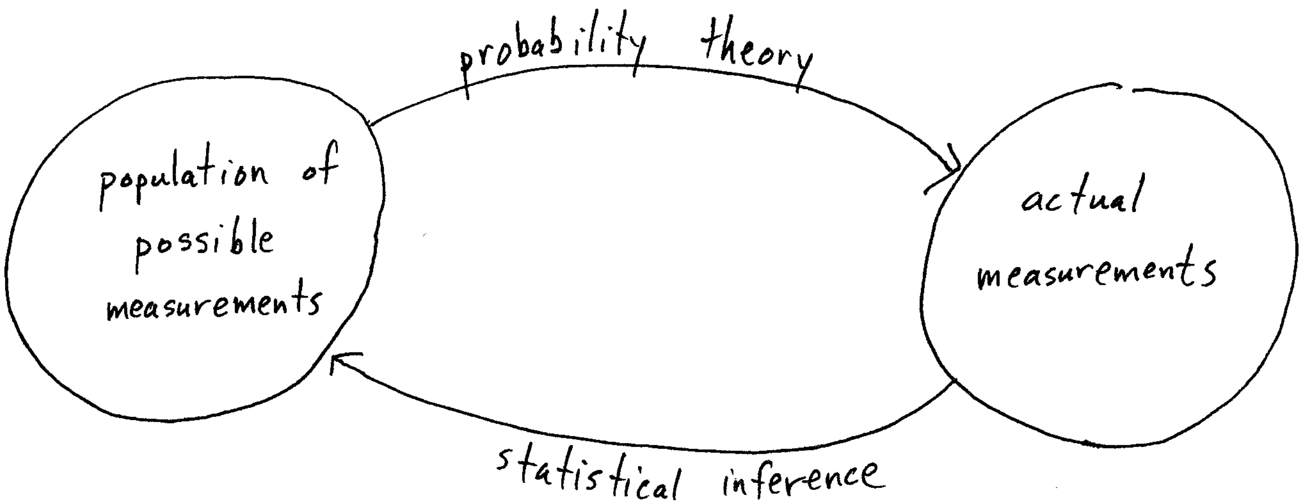
__Statistics__ deals with the collection and analysis of data.

__Data__ is a measured or observed outcome of an experiment or process.

__Probability__ provides a mathematical explanation of the uncertainty/randomness inherent in data.



Successful statisticians must understand both the tools/procedures of statistics and the underlying theory of probability.

Today: Sec. 2.1 Next time: Sec. 2.2

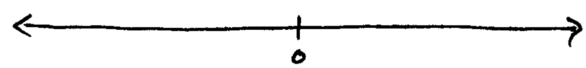
Set Theory

Set theory is the foundation of probability theory.

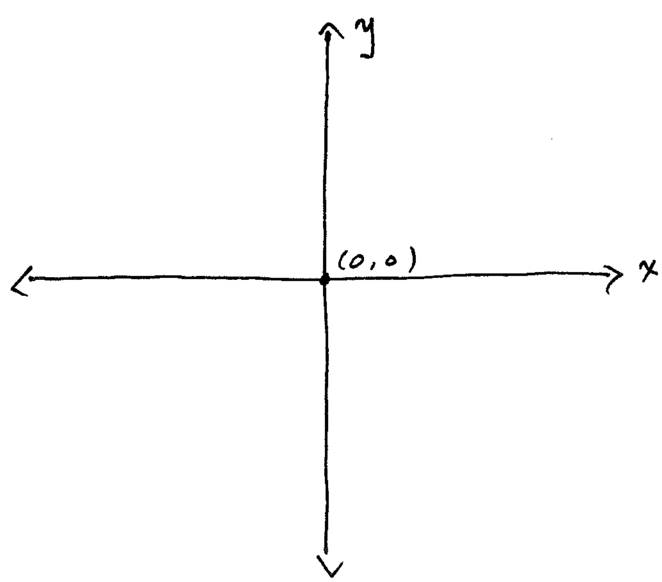
Definitions

- An <u>element</u> is an object or entity of some sort
- A <u>set</u> is a collection of such elements.

Examples

- Positive integers: $\{1, 2, 3, \dots\}$
- Integers: $\{\dots -2, -1, 0, 1, 2, \dots\}$
- Real numbers: $\mathbb{R} =$ 

- $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$



- Primary colors: {red, yellow, blue}

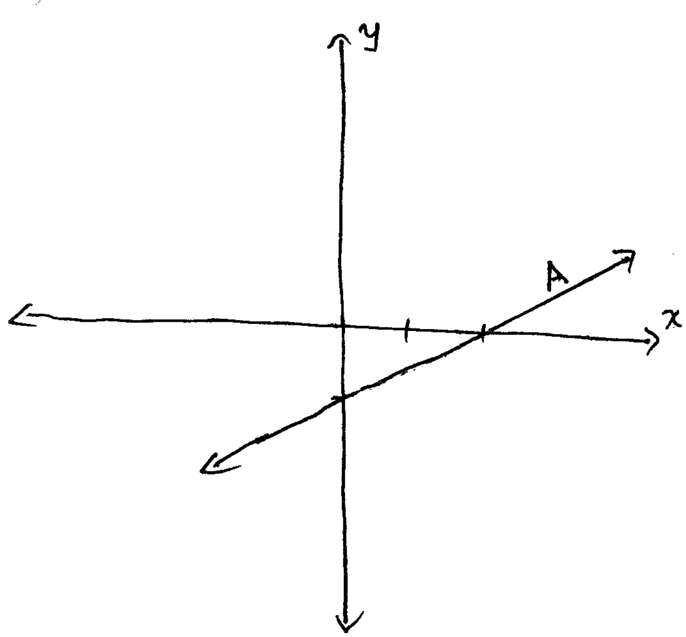
Definitions

- The <u>empty set</u> is the set with no elements, and denoted $\emptyset$.
- If A and B are two sets, and every element of A is an element of B, then A is a <u>subset</u> of B, denoted $A \subset B$.

Examples (of subsets)

- B = all integers
  A = even integers

- $B = \mathbb{R}^2$
  $$A = \{(x,y) : y = \tfrac{1}{2}x - 1\}$$



- B = any set
  $A = \emptyset$

More definitions

- The <u>universal set</u> is the set of all conceivable elements in a given context, and is denoted S.

  [Preview: when we talk about random events, S corresponds to the set of possible outcomes]

Example

If we are concerned with the outcome of the roll of a die, then $S = \{1, 2, 3, 4, 5, 6\}$.

Notation

If x is an element belonging to a set A, we write $x \in A$. If x does not belong to A, we write $x \notin A$.

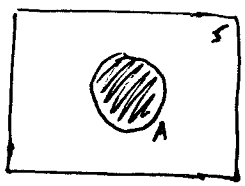
Set Operations

<u>Union</u> $A \cup B := \{x \in S : x \in A \text{ or } x \in B\}$

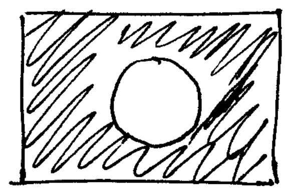
<u>Intersection</u> $A \cap B := \{x \in S : x \in A \text{ and } x \in B\}$

<u>Compliment</u> $A' := \{x \in S : x \notin A\}$

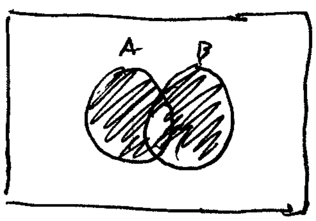
<u>Venn diagrams</u>



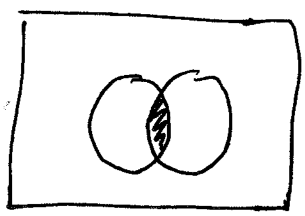
A



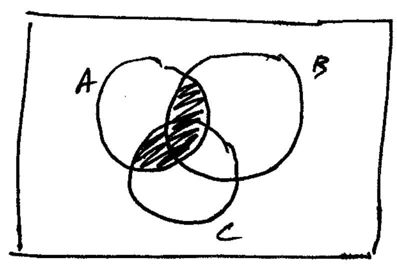
A'



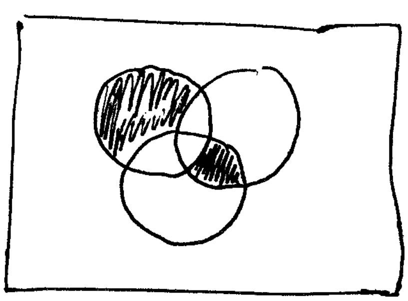
$A \cup B$



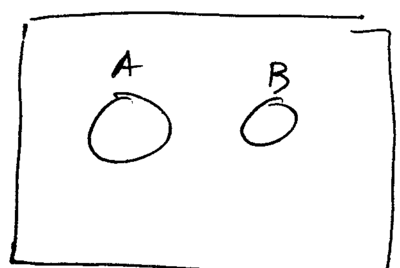
$A \cap B$



$A \cap (B \cup C)$



$(A \cap (B \cup C)') \cup (B \cap C \cap A')$



disjoint.

<u>Terminology</u>: • If $A \cap B = \phi$, then A & B are called <u>disjoint</u>

Properties

- $(A \cup B)' = A' \cap B'$
- $(A \cap B)' = A' \cup B'$
- $\emptyset' = S$
- $S' = \phi$
- $A \cap A' = \phi$
- $A \cup A' = S$
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

- $A \cup \phi = A$
- $A \cap \phi = \phi$
- $A \cup S = S$
- $A \cap S = S$

Probability Theory

Definitions

- The <u>outcome space</u> or <u>sample space</u> of an experiment, denoted S, is the set of possible outcomes.

- An <u>event</u> is a collection of outcomes, i.e., a subset of S.

- If x is measured and $x \in A$, we say event A <u>has occurred</u>.

Examples

- Consider a six-sided die

  $$S = \{1, 2, 3, 4, 5, 6\}$$

  $$A = \{1, 3, 5\}$$

- Deck of cards

  $$S = \{\heartsuit A, \dots, \heartsuit K, \dots, \spadesuit A, \dots, \spadesuit K\}$$

  $$A = \{\text{face cards}\}$$

A probability function assigns probabilities to events, and specifies the chance of different events occurring.

The Axioms of Probability

<u>Definition</u>: A <u>probability function</u> is a mapping P of events $A \subseteq S$ to real numbers such that

(a) $P(A) \geq 0$

(b) $P(S) = 1$

(c) If $A_1, A_2, \ldots$ are mutually exclusive, i.e., $A_i \cap A_j = \phi$ for $i \neq j$, then

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

define

<u>Note</u>: probability functions are always defined w.r.t a sample space S.

<u>Note</u>: this set of axioms is minimal. All intuitive properties can be derived from them, but not if one is omitted.

<u>Example</u> 6-sided die. Define $P(A) = \frac{|A|}{6}$

where $|A| =$ number of elements in A.

(a) ✓ $(|A| \geq 0)$

(b) ✓ $\left(P(S) = \frac{|S|}{6} = \frac{6}{6} = 1\right)$

(c) $|\cup A_i| = \Sigma |A_i|$ when $\{A_i\}$ are mutually exclusive. ✓

Properties

<u>Proposition</u>: For any event A, $P(A) = 1 - P(A')$.

Proof:
$$\begin{aligned} 1 &= P(S) \\ &= P(A \cup A') \\ &= P(A) + P(A') \qquad \text{since } A \cap A' = \phi \end{aligned}$$

$$\Longrightarrow P(A) = 1 - P(A')$$

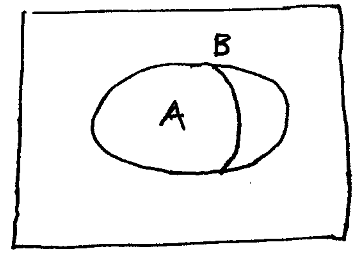
<u>Proposition</u>: $P(\phi) = 0$.

Proof: Apply previous result with $A = \phi$, $A' = S$.

<u>Proposition</u>: If $A \subset B$, then $P(A) \leq P(B)$.

Proof: $B = A \cup (B \cap A')$

and $A \cap (B \cap A') = \phi$



so $P(B) = P(A) + \underbrace{P(B \cap A')}_{\geq 0}$

<u>Prop</u> For any event A, $P(A) \leq 1$.

Proof: Apply previous result with $B = S$.

<u>Prop</u>: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Proof: see book.

Equally Likely Outcomes

A probability set function defines equally likely outcomes when $P(\{x\}) = \frac{1}{|S|}$ for every $x \in S$. By axiom (c) we have $P(A) = \frac{|A|}{|S|}$ as the general probability function.

Example Roll a fair 6-sided die twice. What is the probability that the second roll exceeds the first?

Sol. "fair" connotes equally likely outcomes.

$$S = \{ (i,j) : 1 \le i,j \le 6 \}$$

$$= \{ (1,1), (1,2), (1,3), (1,4), (1,5), (1,6),$$
$$(2,1), (2,2), (2,3), (2,4), (2,5), (2,6),$$
$$\vdots$$
$$(6,1), (6,2), (6,3), (6,4), (6,5), (6,6) \}.$$

$$A = \{ (i,j) : (i,j) \in S, \ i < j \}$$

Then $P(A) = \dfrac{|A|}{|S|} = \dfrac{15}{36} = \dfrac{5}{12}$

Remark Equally likely outcomes are impossible if S is infinite.