$\boxed{\text{Common Discrete Distributions}}$

Let X denote a discrete RV with range S.

~~When~~ Recall a pmf is any function f such that

(a) $f(x) > 0$ for all $x \in S$

(b) $\sum_{x \in S} f(x) = 1$

~~(c) $P(X \in A) = \sum_{x \in A} f(x)$ for all $A \subset \Omega$~~

~~So far we have seen the~~

We will study pmfs with convenient mathematical expressions that arise in many common setting.

<u>So far</u>

Uniform:

$$S = \{1, \ldots, m\}$$

$$f(x) = \frac{1}{m}$$

Bernoulli Trials

A Bernoulli trial is an experiment with only two outcomes.

Examples :

heads or tails
male or female
life or death
pass or fail
success or failure
⋮

A ~~B~~ random variable X such that $X = 0$ or $X = 1$ in accordance w/ the outcome of the Bernoulli trial is ~~called a Bernoulli~~ said to have a Bernoulli distribution.

If p is the probability of success, and $X(\text{success}) = 1$, then the pmf of X is

$$f(1) = p, \quad f(0) = 1 - p.$$

More concisely, we may write

$$f(x) = p^x (1-p)^{1-x} \quad , \quad x = 0 \text{ or } 1$$

Ex ~~In 2004, Albert Pujols had a batting average of .331. ~~Thus, every "at bat" was a Bernoulli trial~~ Therefore, it is reasonable to assume his next at bat will be Bernoulli trial with $p = .331$.~~

Ex: Unbiased coin, lottery

The mean of a Bernoulli R.V. is

$$\mu = E[X] = 0 \cdot f(0) + 1 \cdot f(1)$$
$$= 0 \cdot (1-p) + 1 \cdot p = p.$$

$$\sigma^2 = E[(X-p)^2] = (0-p)^2 \cdot (1-p) + (1-p)^2 \cdot p$$
$$= p^2(1-p) + (1-p)^2 p$$
$$= p(1-p)(p + 1-p)$$
$$= p(1-p)$$

<u>Def</u> • A <u>sequence of Bernoulli trials</u> is when a Bernoulli trial is performed several times, independently, and the prob. of success is constant at p.

- If X is the number of successes in a sequence of n Bernoulli trials, X is said to have a <u>binomial</u> distribution.

Notation: $X \sim \text{Binom}(n, p)$ or $b(n,p)$ $\quad$ n, p = parameters

<u>Example</u>: A biased coin with $p = P(\text{heads})$ is flipped n times. The total number of heads observed is a binomial random variable.

<u>Note</u>: if $n = 1$, then X is simply a Bernoulli R.V.

What is the pmf of $X \sim b(n,p)$?

$$f(x) = P\left(x \text{ successes, } n-x \text{ failures, in some configuration}\right)$$

$$= \binom{n}{x} p^x (1-p)^{n-x}$$

Why? There are $\binom{n}{x}$ ways to position the x successes (i.e., $\binom{n}{x}$ distinguishable permutations)

For each such configuration, since the trials are independent, we may multiply the probabilities to get $p^x (1-p)^{n-x}$.

Summing over the $\binom{n}{x}$ disjoint configurations gives the answer.

<u>Remark</u>: If $f(x)$ is indeed a valid pmf, it should satisfy $\sum_{x=0}^{n} f(x) = 1$. Let's check.

By the binomial formula (see chap. 2),

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

Let $a = p$ and $b = 1-p$. Then

$$\begin{array}{ccl} (p + (1-p))^n & = & \displaystyle\sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x} \\ \| & & \\ 1^n & = & \displaystyle\sum_{x=0}^{n} f(x) \\ \| & & \\ 1 & & \end{array}$$

✓

<u>Example</u>: Suppose you draw a card at random from a deck of 52 <u>with replacement</u>. You do this 9 times. What is the prob. of drawing

(a) 4 clubs? ~~red cards (hearts or diamonds)~~

(b) At least 2 Kings?

(c) More red cards than black cards?

<u>Exercises</u> (don't worry about ~~reducing~~ simplifying expression)

(a) $X = \#$ of clubs

$X \sim b(9, 1/4)$

$P(X=4) = f(4) = \binom{9}{4} \cdot \left(\frac{1}{4}\right)^4 \left(\frac{3}{4}\right)^5 = 0.117$

(b) $X = \#$ of Kings

$X \sim b(9, 1/13)$

$P(X \geq 2) = 1 - P(X=1 \text{ or } X=0)$

$= 1 - f(0) - f(1)$

$= 1 - \binom{9}{0}\left(\frac{1}{13}\right)^0\left(\frac{12}{13}\right)^9 - \binom{9}{1}\left(\frac{1}{13}\right)^1\left(\frac{12}{13}\right)^8$

$= 0.149$

(c) $X = \#$ of red cards

$X = b(9, \frac{1}{2})$

$P(X \geq 5) = \sum_{x=5}^{9} \binom{9}{x}\left(\frac{1}{2}\right)^x\left(\frac{1}{2}\right)^{9-x}$

$= \frac{1}{2}$

If $X \sim b(n,p)$, then

$$\mu_X = np$$
$$\sigma_X^2 = np(1-p).$$

This is proved using the <u>moment generating function</u>
~~next week~~

We would need to evaluate

$$\mu = \sum_{x=0}^{n} x \binom{n}{x} p^x (1-p)^{n-x}$$

Not easy!

Moment Generating Function

<u>Def</u> Let X be a discrete R.V. with range S and pmf $f(x)$. If the series

$$E[e^{tX}] = \sum_{x \in S} e^{tx} f(x)$$

converges for all $t \in [-h, h]$, for some $h > 0$, then $M(t) = E[e^{tX}]$ is called the <u>moment generating function</u> of X.

Fact: If X and Y have the same MGF on some interval $[-h, h]$, then X & Y have the same distribution

Examples (1) If X takes values in $S = \{1, 2, 3\}$, and $f(1) = \frac{1}{3}$, $f(2) = \frac{2}{9}$, $f(3) = \frac{4}{9}$, then

$$M_X(t) = \frac{1}{3}e^t + \frac{2}{9}e^{2t} + \frac{4}{9}e^{3t}$$

(2) $X \sim b(n, p)$. Then

$$M(t) = E\left[e^{Xt}\right] = \sum_{x=0}^{n} e^{xt} f(x)$$

$$= \sum_{x=0}^{n} e^{xt} \binom{n}{x} p^x (1-p)^{n-x}$$

$$= \sum_{x=0}^{n} \binom{n}{x} (pe^t)^x (1-p)^{n-x}$$

Recall: $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$

$$= \left[pe^t + (1-p)\right]^n$$

(3) $S = \{1, 2, 3, \dots\}$

$$f(x) = \frac{1}{2^x}$$

$$M(t) = \sum_{x=1}^{\infty} e^{xt} \cdot \frac{1}{2^x} = ?$$

Aside What is the general formula for

$$Q = \sum_{r=a}^{b} q^r \quad ?$$

Derivation

$$Q = q^a + q^{a+1} + \cdots + q^{b-1} + q^b$$

$$q \cdot Q = q^{a+1} + q^{a+2} + \cdots + q^b + q^{b+1}$$

$$\Rightarrow \underbrace{Q - q \cdot Q}_{= Q(1-q)} = q^a - q^{b+1}$$

$$\Rightarrow Q = \frac{q^a - q^{b+1}}{1-q}, \quad q \neq 1.$$

If $b \to \infty$, and $|q| < 1$, then $q^{b+1} \to 0$

$$\Rightarrow \sum_{r=a}^{\infty} q^r = \frac{q^a}{1-q}.$$

Back to the example:

$$M(t) = \sum_{x=1}^{\infty} e^{xt} \cdot \frac{1}{2^x} = \sum_{x=1}^{\infty} \left(\frac{e^t}{2}\right)^x$$

$$= \frac{(e^t/2)}{1-(e^t/2)} \quad \text{if } \left|\frac{e^t}{2}\right| < 1$$

$$\Updownarrow$$

$$t < \underbrace{\ln 2}_{h}$$

(4) If $M(t) = \left(\frac{2}{3^6}\right) \cdot \frac{(e^t/3)^7}{1-(e^t/3)}$, $t < \ln 3$

what is $f(x)$?

Answer: $f(x) = \frac{2}{3^{x-6}}$, $x = 7, 8, 9, \ldots$

The MGF can be used to generate moments (hence the name).

<u>Definition</u> Let X be a discrete R.V.. The quantity $E[X^r]$, where r is a positive integer, is called the rth moment of X. $\left(= \sum_{x \in S} x^r f(x) \right)$

<u>Ex</u>

- The mean is the 1st order moment.
- The variance can be expressed in terms of the 1st & 2nd order moments:

$$Var(X) = E[(X-\mu)^2] = \cancel{E[(X - E[X}$$

$$= E[X^2 - 2X\mu + \mu^2]$$

$$= E[X^2] - E[2X\mu] + E[\mu^2]$$

$$= E[X^2] - 2\mu E[X] + E[\mu^2]$$

$$= E[X^2] - 2E[X]E[X] + E[E[X]^2]$$

$$= E[X^2] - 2E[X]^2 + E[X]^2$$

$$= E[X^2] - E[X]^2$$

$$M(t) = \sum_{x \in S} e^{tx} f(x)$$

$$M'(t) = \sum_{x \in S} x e^{tx} f(x)$$

$$M''(t) = \sum_{x \in S} x^2 e^{tx} f(x)$$

$$\vdots$$

$$M^{(r)}(t) = \sum_{x \in S} x^r e^{tx} f(x).$$

Evaluate at $t = 0$:

$$M'(0) = \sum x f(x) = E[X]$$

$$M''(0) = \sum x^2 f(x) = E[X^2]$$

$$\vdots$$

$$M^{(r)}(0) = \sum x^r f(x) = E[X^r]$$

<u>Return to binomial example</u>

Recall $X \sim b(n, p)$

$$\Rightarrow M(t) = [(1-p) + pe^t]^n, \quad -\infty < t < \infty$$

$$\Rightarrow M'(t) = n[(1-p) + pe^t]^{n-1}(pe^t)$$

$$\Rightarrow M''(t) = n(n-1)[(1-p) + pe^t]^{n-2}(pe^t)^2$$

$$+ \; n[(1-p) + pe^t]^{n-1}(pe^t)$$

Plug in $t = 0$:

$$\overset{= E[X]}{\mu} = M'(0) = n[(1-p) + p]^{n-1} \cdot p = \underline{\underline{np}}$$

$$E[X^2] = M''(0) = n(n-1)p^2 + np$$

$$= n^2p^2 - np^2 + np$$

$$\Rightarrow \overset{= Var(X)}{\sigma^2} = E[X^2] - E[X]^2$$

$$= n^2p^2 - np^2 + np - (np)^2$$

$$= np - np^2 = \underline{\underline{np(1-p)}}$$

Negative Binomial

Def. Let r be a positive integer. Let X be the number of independent Bernoulli trials needed to obtain r successes. We say X has a negative binomal distribution.

Notation If $p = P(\text{success})$, we write

$$X \sim nb(r, p).$$

What is the pmf? By the multiplication rule (for independent events)

$$f(x) = P(X = x) =$$

$$P(r-1 \text{ successes in } x-1 \text{ trials}) \cdot P(\text{success on } x^{th} \text{ trial})$$

$$= \binom{x-1}{r-1} p^{r-1} (1-p)^{x-1-(r-1)} \cdot p$$

$$= \binom{x-1}{r-1} p^{r} (1-p)^{x-r}, \quad x = r, r+1, \ldots$$

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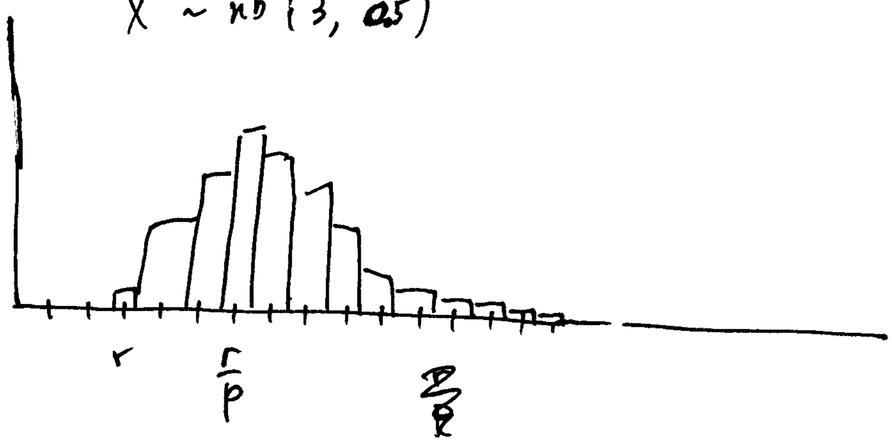
From the book

$$M(t) = \frac{(pe^t)^r}{[1-(1-p)e^t]^r}, \quad (1-p)e^t < 1$$

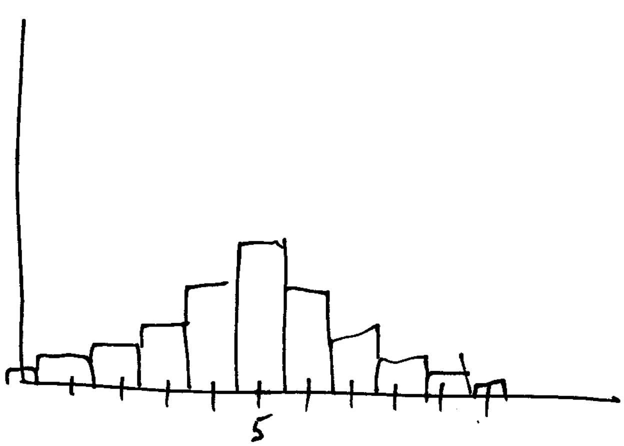
$$\mu = \frac{r}{p}, \quad \sigma^2 = \frac{r(1-p)}{p^2}$$

Visually

$X \sim nb(3, .5)$



$X \sim b(10, .5)$



Hypergeometric

Suppose there are 2 types of objects, N_1 of one type, N_2 of the other. ~~n of these o~~ The objects are pooled and n are choosen at random. Let X be the number drawn ~~from type~~ that are of type 1. We say X has a hypergeometric distribution.

Notation $\quad X \sim hg(N_1, N_2, n)$

~~Assum~~ PMF: $\quad f(x) = P(X = x)$

$$= \frac{N(x \text{ of type } 1,\ n-x \text{ of type } 2)}{N(n \text{ total})}$$

$$= \frac{N(x \text{ of type } 1) \cdot N(n-x \text{ of type } 2)}{N(n \text{ total})}$$

$$= \binom{N_1}{x} \cdot \binom{N_2}{n-x} \Big/ \binom{N_1 + N_2}{n}, \quad \begin{array}{l} x \le n \\ x \le N_1 \\ n-x \le N_2 \end{array}$$

Mean & Variance (Appendix A.2)

$$\mu = n \cdot \frac{N_1}{N_1+N_2}$$

$$\sigma^2 = n \cdot \left(\frac{N_1}{N_1+N_2}\right) \cdot \left(\frac{N_2}{N_1+N_2}\right) \left(\frac{N_1+N_2-n}{N_1+N_2-1}\right)$$

Poisson

A random variable has a Poisson distribution if its p.m.f. is

$$f(x) = \frac{e^{-\lambda} \cdot \lambda^x}{x!}, \quad x = 0, 1, 2, \ldots$$

for some real number $\lambda > 0$.

Verify valid pmf:

$$\sum f(x) = \sum e^{-\lambda} \frac{\lambda^x}{x!}$$

$$= e^{-\lambda}\left(1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots\right)$$

$$= e^{-\lambda} e^{\lambda} = 1 \quad \text{(MacLaurin series)}$$

Notation $X \sim \text{poisson}(\lambda)$

Motivation (Counting processes) Suppose a random event happen λ times on average in a given unit of time.

$f(x)$ is the prob. that the event will occur exactly x times. (Derivation in book)

Example Let X denote the number of

~~Suppose~~ calls that arrive at a switchboard in

a minute. If an average of 3 calls

arrive per minute, then $X \sim \text{Poisson}(3)$

and $P(X \le 5) = \sum_{X=0}^{5} e^{-5} \cdot \frac{5^x}{x!} = 0.92$

$$\boxed{\text{Remark: Probability Models}}$$

A probability <u>model</u> is a pmf (or pdf) that one hopes is a reasonable appoximation to ~~a~~ the true distribution of a random variable.

The Poisson distrib. is a common <u>model</u> for counting processes.

<u>Q</u>: Can you think of other examples of Poisson RVs?

<u>A</u>: planes arriving @ airport, packet in network switch, α particles hitting Geiger counter.

In reality, most counting processes are not ~~truely~~ truly Poisson distributed. For a ~~true~~ Poisson process arbitrarily large outcomes have nonzero prob, but in reality physical contraints prevent this. Nonetheless, the Poisson is often a useful model in practice because the discrepancies with reality occur with extremely small probability.

Exercise Find $M(t)$, μ, σ^2 if $X \sim \text{Poisson}(\lambda)$

Sol: $M(t) = E[e^{tX}] = \sum_{x=0}^{\infty} e^{tx} e^{-\lambda} \frac{\lambda^x}{x!}$

$= \sum_{x=0}^{\infty} e^{-\lambda} \frac{(\lambda e^t)^x}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!} = e^{-\lambda} e^{\lambda e^t}$

$= e^{\lambda(e^t - 1)}$ ∎.

$M'(t) = \lambda e^t e^{\lambda(e^t-1)} \Rightarrow \mu = M'(0) = \lambda$

$M''(t) = (\lambda e^t)^2 e^{\lambda(e^t-1)} + \lambda e^t e^{\lambda(e^t-1)} \Rightarrow E[X^2] = M''(0) = \lambda^2 + \lambda$

$\Rightarrow \sigma^2 = E[X^2] - E[X]^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$

Binomial Approximation

Fact: If $X \sim \text{Poisson}(\lambda)$, then

$$f(x) = P(X = x) \approx \binom{n}{x}\left(\frac{\lambda}{n}\right)^x\left(1 - \frac{\lambda}{n}\right)^{n-x}$$

That is, $X \sim b\left(n, \frac{\lambda}{n}\right)$ approximately.

The approximation is good if n is large.

Thus, the poisson pmf may be used to approximate the binomial and vice-versa.