Conditional Probability

What is the probability of event A, given that event B has occurred? Notation: $P(A/B)$

Example Suppose a fair 6-sided die is rolled.

What is the probability that the roll is odd, given that the roll is ≤ 3?

$$S = \{1, 2, 3, 4, 5, 6\}$$

$$A = \{1, 3, 5\}$$

$$B = \{1, 2, 3\}$$

Intuitively we expect $P(A/B) = \frac{2}{3}$, whereas $P(A) = \frac{1}{2}$ without knowledge of B.

Observe

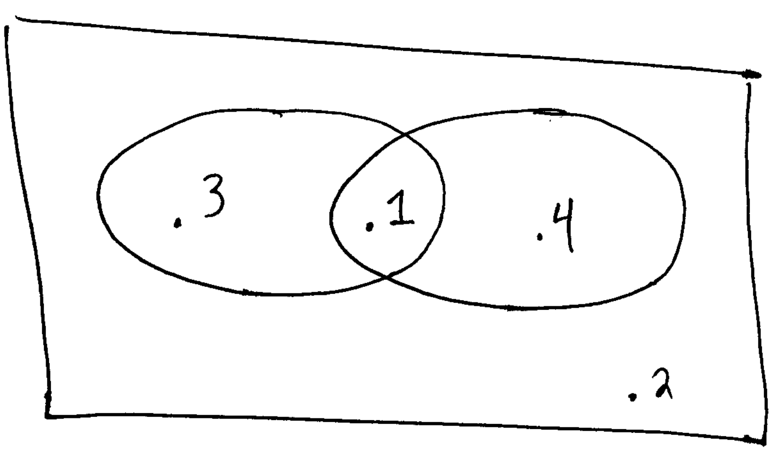
$$\frac{2}{3} = \frac{|A \cap B|}{|B|} = \frac{|A \cap B| / |S|}{|B| / |S|} = \frac{P(A \cap B)}{P(B)}.$$

<u>Definition</u> Let B denote an event w/ $P(B) > 0$. The conditional probability of event A given that event B has occurred is defined by

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

<u>Venn Diagrams</u>

Suppose $P(A) = .3$, $P(B) = .4$, and $P(A \cup B) = .8$. What is $P(A|B)$? $P(A|B')$?



$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A) + P(B) - P(A \cup B)}{P(B)} = \frac{.1}{.4} = \frac{1}{4}$$

Exercise

$$P(A|B') = \frac{P(A \cap B')}{P(B')} = \frac{P(A) - P(A \cap B)}{1 - P(B)} = \frac{.2}{.6} = \frac{1}{3}$$

Remark For a fixed B with $P(B) > 0$, the operator $P(\cdot \mid B)$ is also a valid probability function on the ~~sample~~ "effective" sample space B.

(a) $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)} \geq 0$

(b) $P(B \mid B) = \dfrac{P(B \cap B)}{P(B)} = \dfrac{P(B)}{P(B)} = 1$

(c) $$P\left(\bigcup_{i=1}^{\infty} A_i \mid B\right) = \frac{P\left(\left(\bigcup_{i=1}^{\infty} A_i\right) \cap B\right)}{P(B)}$$

$$= \frac{P\left(\bigcup_{i=1}^{\infty} (A_i \cap B)\right)}{P(B)}$$

$$= \frac{\sum_{i=1}^{\infty} P(A_i \cap B)}{P(B)}$$ [assume the A_i are disjoint]

$$= \sum_{i=1}^{\infty} P(A_i \mid B)$$

Example In the previous example,

$$P(A' \mid B) = 1 - P(A/B) = \frac{3}{4}$$

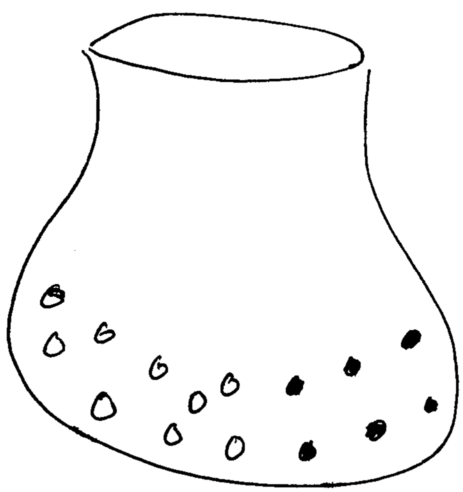
$$P(A' \mid B') = 1 - P(A/B') = \frac{2}{3}$$

The Multiplication Rule

Sometimes it is easy to specify $P(A/B)$, which gives the following formula for $P(A \cap B)$:

$$P(A \cap B) = P(A) \cdot P(B/A).$$
$$= P(B) \cdot P(A/B)$$

Example



Draw balls from an urn w/o replacement

9 white
6 black

A = first ball is black
B = 2nd " " white

$$P(A \cap B) = P(A) \cdot P(B \mid A) = \frac{6}{15} \cdot \frac{9}{14} = \frac{9}{35}$$

This rule (the mult. rule) extends to multiple events: For example

If A, B, C are 3 events, then

$$P(A \cap B \cap C) = P(A) \cdot P(B|A) \cdot P(C|A \cap B)$$

<u>Proof</u>:

$$= P((A \cap B) \cap C)$$

$$= P(A \cap B) \cdot P(C \mid A \cap B)$$

$$= P(A) \cdot P(B|A) \cdot P(C \mid A \cap B) \qquad \blacksquare$$

<u>Example</u> 4 cards are drawn at random, w/o replacement, from a deck of 52. What is the probability of receiving, in order, a spade, a heart, a diamond, & a club?

$$P(A \cap B \cap C \cap D) = \frac{13}{52} \cdot \frac{13}{51} \cdot \frac{13}{50} \cdot \frac{13}{49}$$