Confidence Intervals for Means

Suppose $X_1, \dots, X_{30} \overset{IID}{\sim} N(\mu, 1)$

and μ is unknown. We would like to

a) estimate μ

b) say something about how close our estimate is to the true value of μ.

As an estimate, let's consider

$$\bar{X} = \frac{1}{n} \sum_{i=1}^{30} X_i$$

Recall $\bar{X} \sim N(\mu, \frac{1}{30})$

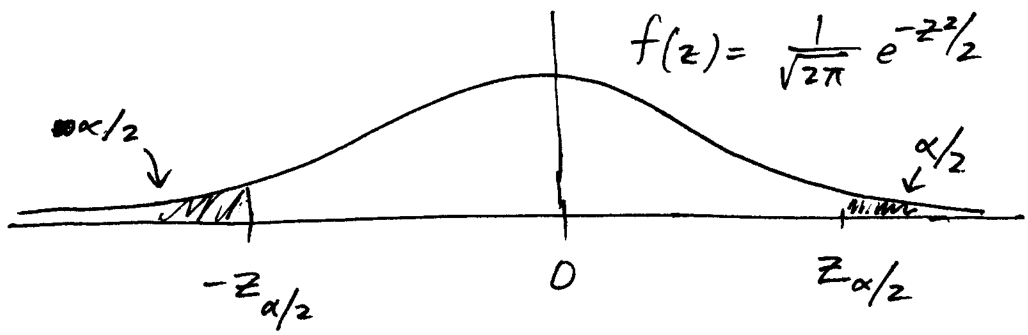
and

$$\frac{\bar{X} - \mu}{\sqrt{1/30}} \sim N(0, 1)$$

Also recall that if $Z \sim N(0,1)$ then

$$P(Z \leq z_\alpha) = 1 - \alpha$$

$$\cancel{P(Z \geq -Z_\alpha) = 1-\alpha}$$



$$\Rightarrow \quad P\left(-z_{\alpha/2} \le Z \le z_{\alpha/2}\right) = 1 - \alpha$$

Take $\alpha = 0.1$. Then $z_{0.05} = 1.645$.

Therefore

$$P\left(-1.645 \le \frac{\bar{X} - \mu}{\sqrt{1/30}} \le 1.645\right) = .9$$

But $-1.645 \le \dfrac{\bar{X} - \mu}{\sqrt{1/30}} \le 1.645$

$$\Updownarrow$$

$$-1.645\left(\sqrt{\tfrac{1}{30}}\right) - \bar{X} \le -\mu \le -\bar{X} + 1.645\left(\sqrt{\tfrac{1}{30}}\right)$$

$$\Updownarrow$$

$$-.3 - \bar{X} \le -\mu \le -\bar{X} + .3$$

$$\Updownarrow$$

$$\bar{X} + .3 \ge \mu \ge \bar{X} - .3$$

$$\Updownarrow$$

$$\bar{X} - .3 \le \mu \le \bar{X} + .3$$

Hence,

$$P\left(\bar{X} - .3 \le \mu \le \bar{X} + .3 \right) = 0.9$$

The probability that the true mean μ is between $\bar{X} - .3$ and $\bar{X} + .3$ is 0.9

The interval $\bar{X} \pm 0.3$ is called the 90% confidence interval for μ.

<u>More generally</u>: Let $X_1, \ldots, X_n \overset{IID}{\sim} N(\mu, \sigma^2)$ with n, σ^2 known, μ unknown.

Let $\alpha \in (0,1)$. Find c such that

$P\left(\bar{X} - c \le \mu \le \bar{X} + c \right) = 1 - \alpha$. c will depend on σ^2, n, and α.

Solution:

$$P\left(-z_{\alpha/2} \le \frac{\bar{X}-\mu}{\sqrt{\sigma^2/n}} \le z_{\alpha/2}\right) = 1-\alpha$$

$$\Updownarrow$$

$$-z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \le \bar{X}-\mu \le z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

$$\Updownarrow$$

$$\bar{X}-z_{\alpha/2}\left(\frac{\sigma}{\sqrt{n}}\right) \le \mu \le \bar{X}+z_{\alpha/2}\left(\frac{\sigma}{\sqrt{n}}\right)$$

Hence, the <u>$100(1-\alpha)\%$ confidence interval</u> for μ is $\bar{X} \pm z_{\alpha/2}\left(\frac{\sigma}{\sqrt{n}}\right)$.

What happens as n increases? σ? α?

For more examples, see Section 7.2.