Confidence Intervals for means: Unknown Variance

Previously we showed that if

$X_1, \dots, X_n \overset{\text{IID}}{\sim} N(\mu, \sigma^2)$, then

$$P\left(\bar{X} - z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \le \mu \le \bar{X} + z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \right) = 1-\alpha$$

$$\Updownarrow$$

$$P\left(|\bar{X} - \mu| \le z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \right) = 1-\alpha .$$

Thus, $\bar{X} \pm z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$ is <u>a</u> $100(1-\alpha)\%$ confidence interval for μ. What if σ is unknown?

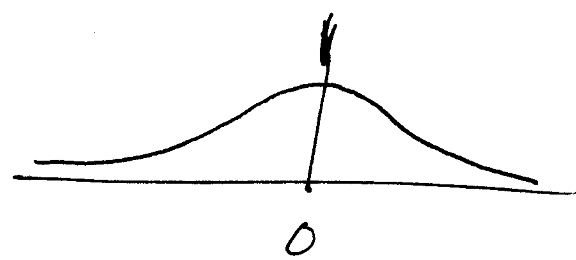
Def ~~Consider~~ The random variable T with pdf

$$f(t) = \frac{\Gamma\left(\frac{r+1}{2}\right)}{\sqrt{\pi r}\,\Gamma\left(\frac{r}{2}\right)\cdot\left(1+\frac{t^2}{r}\right)^{\frac{r+1}{2}}}, \quad -\infty < t < \infty,$$

is ~~called~~ said to have a "t distribution with r deg. freedom"

Notation: $T \sim t_{(r)}$.

Special case: Cauchy $(r=1)$



Theorem: If $Z \sim N(0,1)$, $U \sim \chi^2(r)$, U, Z independent, then

$$T = \frac{Z}{\sqrt{U/r}} \sim t(r).$$

Recall: If $X_1, \ldots, X_n \overset{IID}{\sim} N(\mu, \sigma^2)$, then

$$\left.\begin{aligned} \bar{X} &\sim N\left(\mu, \frac{\sigma^2}{n}\right) \\ \frac{(n-1)S^2}{\sigma^2} &\sim \chi^2(n-1) \end{aligned}\right\} \text{independent.}$$

Now set $T = \dfrac{\bar{X} - \mu}{S/\sqrt{n}}$. We can write

$$T = \frac{\dfrac{\bar{X} - \mu}{\sigma/\sqrt{n}}}{\sqrt{\dfrac{(n-1)S^2}{\sigma^2} \Big/ (n-1)}} = \frac{Z}{\sqrt{U/r}}$$

where

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$$

$$U = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2(r)$$

$$r = n-1$$

(4)

Conclusion:
$$T = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim t_{(n-1)}$$

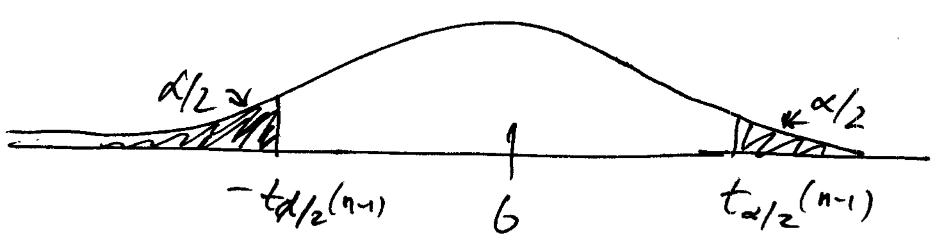
Define $t_\alpha(n-1)$ to be the number such that

$$P\left(T > t_\alpha(n-1)\right) = \alpha$$

where T is a $t(n-1)$ R.V.

Then

$$P\left(-t_{\alpha/2}(n-1) \le \frac{\bar{X}-\mu}{s/\sqrt{n}} \le t_{\alpha/2}\right) = 1-\alpha.$$



Rearranging terms, we have

$$P\left(\bar{X} - t_{\alpha/2}(n-1)\cdot\left(\frac{s}{\sqrt{n}}\right) \le \mu \le \bar{X} + t_{\alpha/2}\left(\frac{s}{\sqrt{n}}\right)\right) = 1-\alpha$$

Hence, $\bar{X} \pm t_{\alpha/2}\left(\frac{s}{\sqrt{n}}\right)$ is a $100(1-\alpha)\%$ confidence interval for μ.