$\boxed{\text{Continuous Random Variables}}$

Recall: A random variable is a function X mapping S_0 (outcomes of some experiment) to $\mathbb{R}$ (the real numbers). The range of X is the set of possible values X can take.
(denoted S)

Two kinds of RVs

(1) Discrete: For any $x_1, x_2 \in S$, there exists a real number $r \notin S$ that is between them.

(2) Continuous: For any $x_1 \in S$, there exists $x_2 \in S$ such at all points between x_1 and x_2 are in S.

Examples

- X = temperature (in Kelvins)

  $S = [0, \infty)$

- X = distance of a javelin toss (in meters)

  $S = [0, 1000]$

Definition Let X be a continuous R.V. with range S. The prob. density function (pdf) for X is a function $f(x)$ such that

(a) $f(x) \geq 0$ for $x \in S$

(b) $$\int_S f(x)\,dx = 1.$$

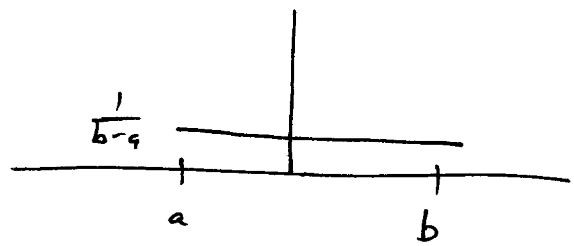
(c) ~~If f is a pdf, then~~ the probability of the event $X \in A$ (where $A \subseteq S$) is

$$P(X \in A) = \int_A f(x)\,dx.$$

<u>Example</u> Uniform RV on $[a, b]$.

$$f(x) = \frac{1}{b-a}, \quad x \in [a, b]$$

$$\geq 0$$



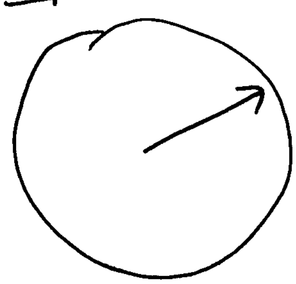
$$\int_S f(x) = \int_a^b \frac{1}{b-a}\, dx$$

$$= \frac{b-a}{b-a} = 1.$$

$\cancel{P\{X \in [r, s]\} =}$

<u>Notation</u> $X = \text{unif}(a, b)$

<u>Spinner</u>



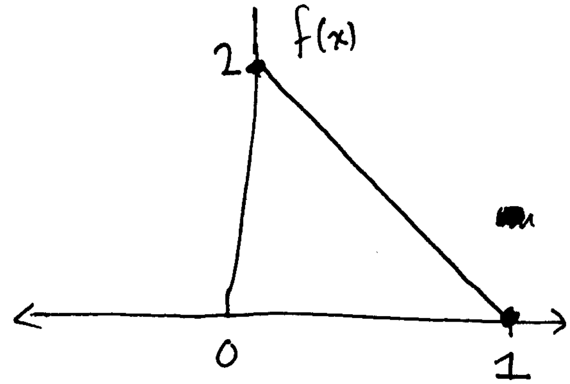
X = angle in radians of arrow after a spin;

$$X \sim \text{unif}(0, 2\pi).$$

$$P\left(X \in \left[\tfrac{\pi}{2}, \tfrac{3}{4}\pi\right]\right) = \frac{1}{2\pi} \int_{\frac{\pi}{2}}^{\frac{3}{4}\pi} dx = \frac{\frac{3}{4} - \frac{1}{2}}{2} = \frac{1}{8}.$$

Example

$X =$ distance a dart lands from the bullseye on a dart board.



$f(x) = 2 - 2x, \quad 0 \le x \le 1.$

$$P\left(X \le \tfrac{1}{4}\right) = \int_0^{\frac{1}{4}} (2-2x)\,dx = \left[2x - x^2\right]_0^{\frac{1}{4}}$$

$$= 2 \cdot \tfrac{1}{4} - \left(\tfrac{1}{4}\right)^2 = \tfrac{1}{2} - \tfrac{1}{16} = \tfrac{7}{16}$$

Convention Extend $f(x)$ to $\mathbb{R}$ by setting

$f(x) = 0$ for $x \notin S$.

Expectation, etc.

If X is a continuous R.V. w/ range S and pdf $f(x)$, and $u(X)$ is a function of X, define

$$E\left[u(X)\right] = \int_{-\infty}^{\infty} u(x) f(x)\, dx.$$

Special cases

- $\mu = E[X] = \int_{-\infty}^{\infty} x f(x)\, dx$
- $\sigma^2 = E[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx$
- r^{th} order moment : $E[X^r] = \int_{-\infty}^{\infty} x^r f(x)\, dx$
- MGF : $M(t) = E\left[e^{tX}\right] = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx,$

  $-h < t < h,$ (when it exists)

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As in the discrete case, the following properties hold:

- $E[a u_1(X) + b u_2(X)] = a E[u_1(X)] + b E[u_2(X)]$

- $\sigma^2 = E[X^2] - E[X]^2$

- $$M(0) = 1$$
  $$M'(0) = E[X]$$
  $$\vdots$$
  $$M^{(r)}(0) = E[X^r]$$

Remark about continuous RVs

- Any event with a finite (or even countable) number of outcomes has probability zero.

- Changing the pdf at a finite (or even countable) # of points does not change the random variable.

Reason: For any $x_0 \in S$,

~~Ex: Uniform~~ $P(\{x_0\}) = \int_{x_0}^{x_0} f(x)\,dx = 0$

- Intuition: prob = area under pdf.

Two Important Families of Distributions

① **Uniform** $a, b \in \mathbb{R}$, $a < b$

$$S = [a, b]$$

$$f(x) = \begin{cases} \dfrac{1}{b-a}, & x \in [a,b] \\ 0 & \text{otherwise} \end{cases}$$

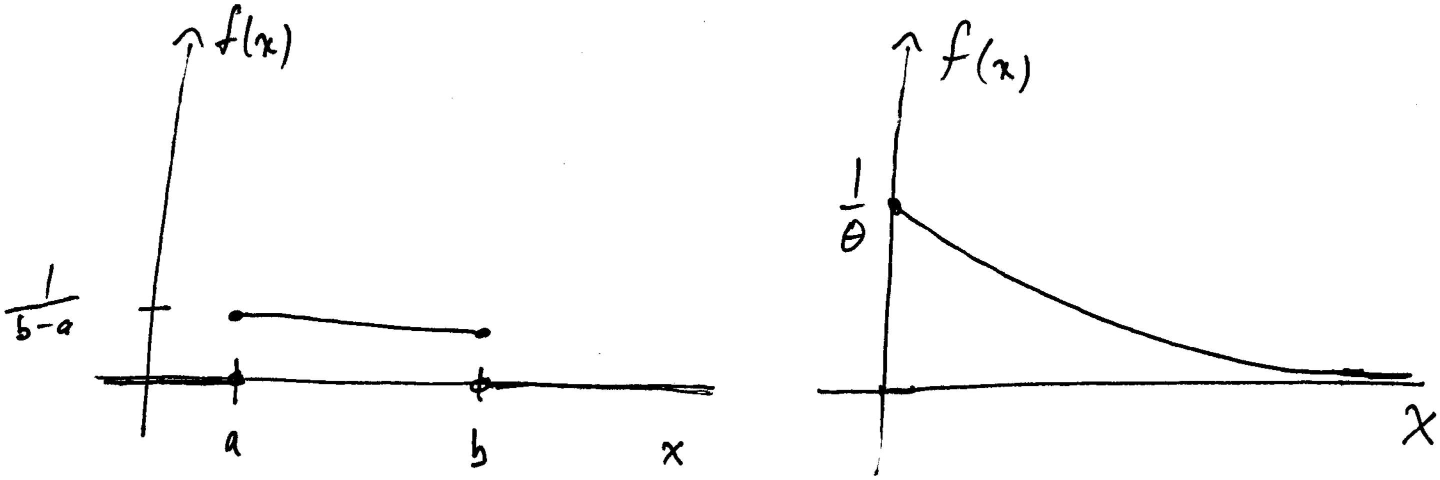
Notation: $X \sim \text{Unif}(a,b)$

② **Exponential** $\theta \in \mathbb{R}$, $\theta > 0$

$$S = [0, \infty)$$

$$f(x) = \begin{cases} \dfrac{1}{\theta} e^{-x/\theta}, & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Notation: $X \sim \exp(\theta)$



Example Suppose $X \sim Unif(a,b)$

$$E[X] = \int_S x f(x) = \int_a^b x \cdot \frac{1}{b-a}\, dx$$

$$= \frac{\frac{1}{2} x^2 \Big|_a^b}{b-a} = \frac{\frac{1}{2}(b^2-a^2)}{b-a} = \frac{(b-a)(b+a)}{2(b-a)}$$

$$= \frac{a+b}{2}$$

$$E[X^2] = \int_a^b x^2 \frac{1}{b-a}\, dx = \frac{1}{b-a}\left[\frac{1}{3}x^3\right]_a^b$$

$$= \frac{b^3-a^3}{3(b-a)} = \frac{(b-a)(b^2+ab+a^2)}{3(b-a)}$$

$$= \frac{b^2+ab+a^2}{3}$$

$$\sigma^2 = \frac{b^2+ab+a^2}{3} - \left(\frac{a+b}{2}\right)^2$$

$$= \frac{(b-a)^2}{12}$$

$$M(t) = \int_S e^{xt} f(x)\, dx = \int_a^b \frac{e^{xt}}{b-a}\, dx$$

$$= \frac{1}{(b-a)t} e^{xt} \Big|_a^b = \frac{e^{bt} - e^{at}}{t(b-a)}, \quad t \neq 0.$$

If $t=0$, $M(0) = \int_a^b f(x) = 1$.

<u>At home Exercise</u> verify μ, σ^2 using MGF

<u>In-class Excercise</u> Compute MGF, μ, σ^2 if $X \sim \exp(\theta)$.

<u>Solution</u>: $f(x) = \frac{1}{\theta} e^{-x/\theta}$, $x \geq 0$

$$M(t) = \int_0^\infty e^{tx} \frac{1}{\theta} e^{-x/\theta}\, dx = \int_0^\infty \frac{1}{\theta} e^{(t-\frac{1}{\theta})x}\, dx$$

$$= \frac{1}{\theta} \left[\frac{1}{(t-\frac{1}{\theta})} e^{(t-\frac{1}{\theta})x} \right]_{x=0}^{x=\infty}$$

$$= -\frac{1}{\theta} \cdot \frac{1}{t-\frac{1}{\theta}} \qquad \left(\text{provided } t < \frac{1}{\theta}\right)$$

$$= \frac{1}{1-\theta t}$$

$$M'(t) = \frac{\theta}{(1-\theta t)^2} \Rightarrow \mu = M'(0) = \boxed{\theta}$$

$$M''(t) = \frac{2\theta^2}{(1-\theta t)^3} \Rightarrow E[X^2] = 2\theta^2$$

$$\Rightarrow \sigma^2 = 2\theta^2 - \theta^2 = \boxed{\theta^2}$$

- Discuss relationship of $\exp(\theta) = \text{Poisson}(\lambda)$, $\theta = 1/\lambda$

<u>Definition</u> Let X be a continuous R.V. with pdf $f(x)$. The <u>cumulative distribution function</u> of X is

$$F(x) = P(X \leq x) = \int_0^x f(y)\,dy$$

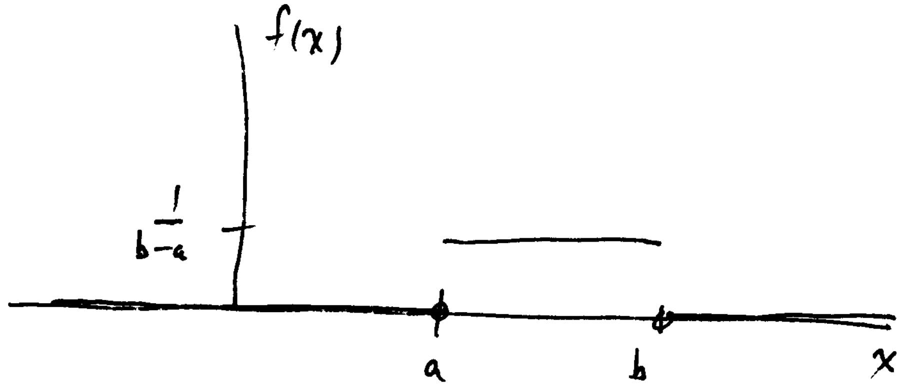
<u>Example</u> $X \sim \text{Unif}(a,b)$, $f(x) = \frac{1}{b-a}$, $a \leq x \leq b$

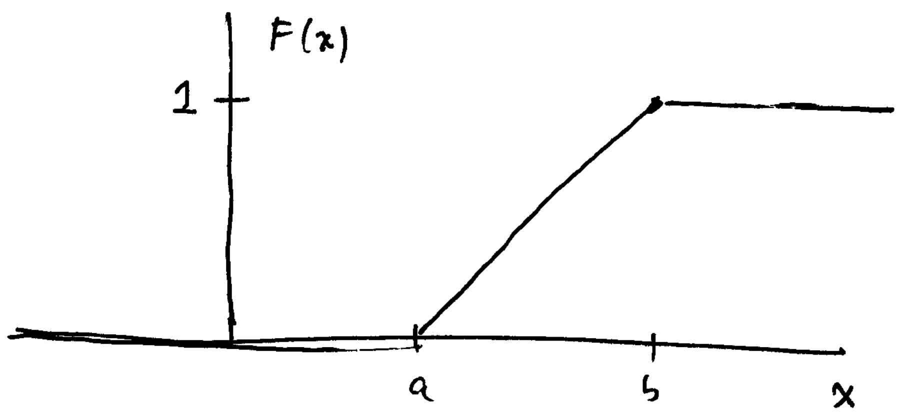
Three cases:

<u>$x < a$</u>: $F(x) = 0$

<u>$a \leq x \leq b$</u>: $F(x) = \int_a^x \frac{1}{b-a} x = \frac{x-a}{b-a}$

<u>$x > b$</u>: $F(x) = 1$.



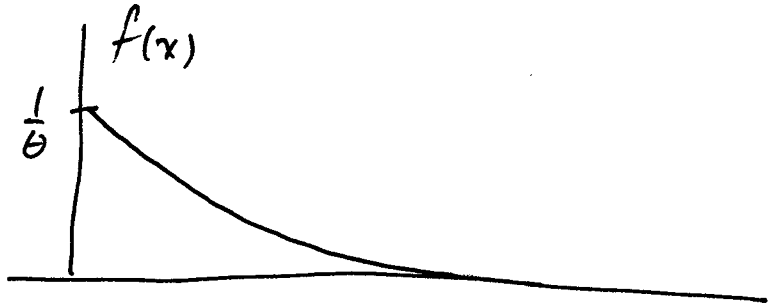


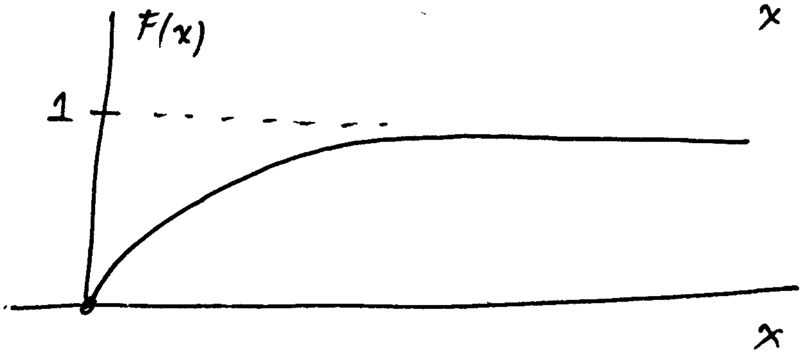
~~Example~~ Exercise Determine $F(x)$ if $X \sim \exp(\theta)$.

Sol $F(x) = 0$ if $x < 0$. If $x \geq 0$, then

$$F(x) = \int_0^x \frac{1}{\theta} e^{-y/\theta}\, dy = -e^{-y/\theta}\Big|_{y=0}^{y=x}$$

$$= 1 - e^{-x/\theta}$$





Properties of CDFs

① $F(-\infty) = \lim_{x \to -\infty} F(x) = 0$

② $F(\infty) = \lim_{x \to \infty} F(x) = 1$

③ $F'(x) = f(x)$ (Fund. Thm. Calc.)

④ $P([r, s]) = \int_r^s f(x)\,dx = F(s) - F(r).$

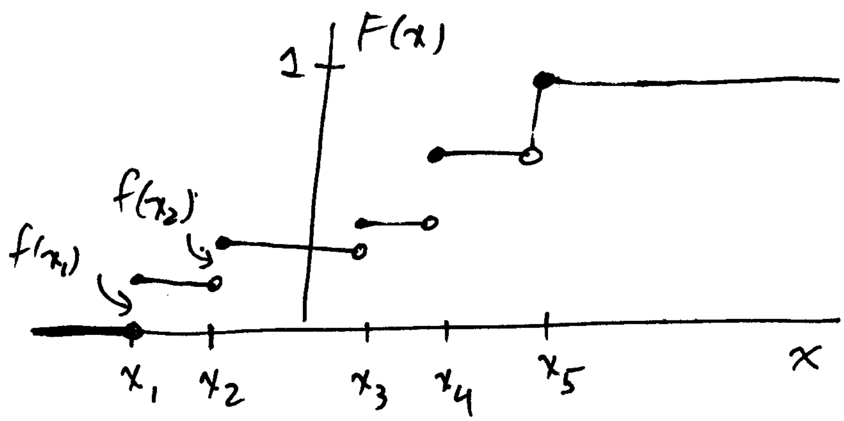
Remark CDFs are also defined for discrete RVs:

$$F(x) = P(X \le x) = \sum_{y \le x} f(x)$$

The difference:

Continuous RV: $F(x)$ is a continuous function

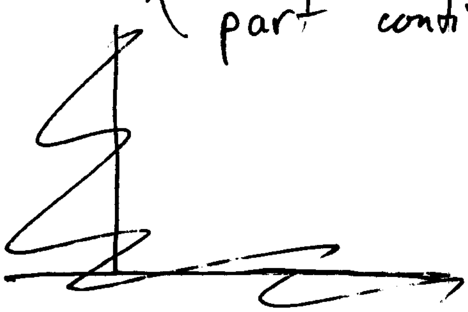
Discrete RV: $F(x)$ is a step function:



~~The~~ **Mixed RV** CMF has jumps and smooth transitions

↑ part continuous, part discrete

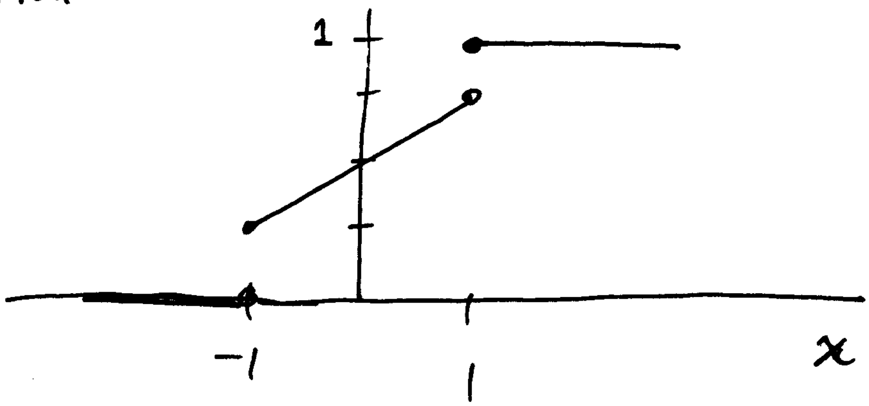
$\boxed{\text{E.g.}}$



$X = \mathrm{Unif}(-2,2)$

$$Y = \begin{cases} -1 & \text{if } X \le -1 \\ X & \text{if } -1 \le X \le 1 \\ 1 & \text{if } X > 1 \end{cases}$$

Then



<u>Percentiles</u> Let $p \in [0,1]$. The

The $(100p)$th <u>percentile</u> is a number

π_p such that

$$F(\pi_p) = \cancel{P(X \le \pi_p)} = p,$$

i.e. $p = F(\pi_p) = P(X \le \pi_p) = \int_{-\infty}^{\pi_p} f(x)\,dx$

<u>Special cases</u>

$p = 0.25$ first quartile

0.5 median

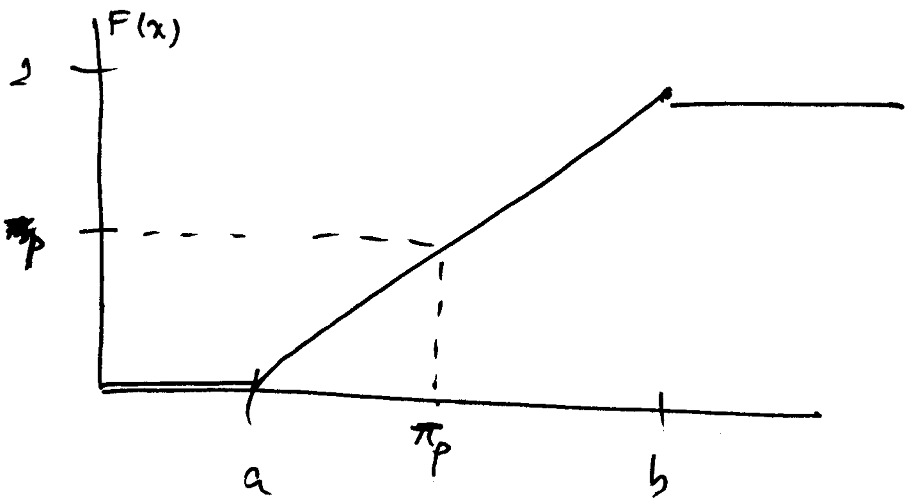
0.75 3^{rd} quartile

<u>Example</u> $X \sim \text{Unif}(a, b)$

$$F(x) = \begin{cases} 0 & x < a \\ \dfrac{x-a}{b-a} & a \le x \le b \\ 1 & x > b \end{cases}$$

$$F(\pi_p) = p \implies p = \frac{\pi_p - a}{b-a}$$

$$\implies \pi_p = (b-a)p + a$$



<u>Ex</u>:

$$p = \tfrac{1}{2} \implies \pi_p = (b-a)\tfrac{1}{2} + a = \frac{b+a}{2}$$

$$p = \tfrac{1}{4} \implies \pi_p = a + \frac{(b-a)}{4}$$

$$p = 0 \implies \pi_p = \text{any number} \le a.$$

Exercise Find π_p as a func. of p if $X \sim \exp(\theta)$.

Recall ~~Pf~~ $F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-x/\theta} & x \geq 0 \end{cases}$

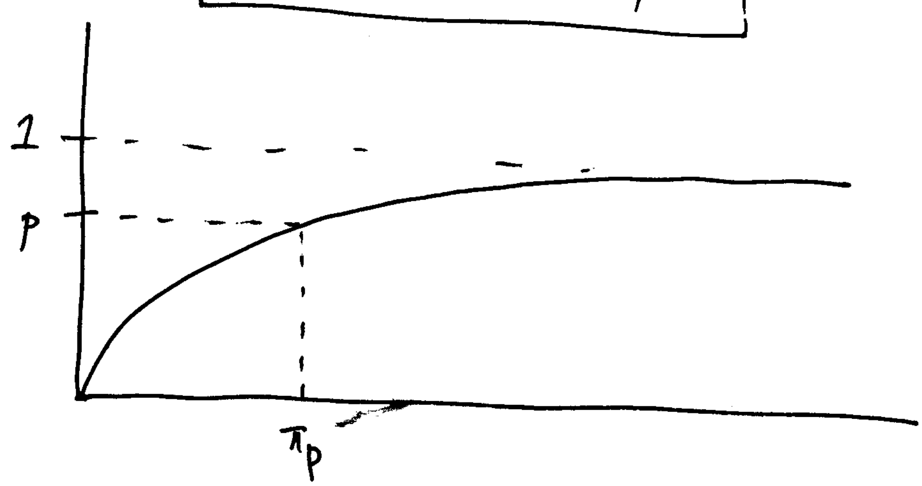
Sol

$$p = F(\pi_p) = 1 - e^{-\pi_p/\theta}$$

$$\Rightarrow e^{-\pi_p/\theta} = 1 - p$$

$$\Rightarrow -\pi_p/\theta = \ln(1-p)$$

$$\Rightarrow \boxed{\begin{aligned} \pi_p &= -\theta \ln(1-p) \\ &= \theta \ln\left(\frac{1}{1-p}\right) \end{aligned}}$$



Median: $\pi_{0.5} = \theta \ln\left(\frac{1}{1-\frac{1}{2}}\right) = \theta \ln(2)$.