Counting Techniques

When computing probabilities it is often necessary to count the total possible # of outcomes and the # of outcomes in a particular event.

Example: A fair 6-sided die is rolled 5 times.

What is the probability that the sum of the rolls is 17?

$$S = \{(r_1, \dots, r_5) : 1 \le r_i \le 6\}$$

$$A_{17} = \{(r_1, \dots, r_5) \in S : \sum r_i = 17\}.$$

$$|S| = 6^5, \quad |A_{17}| = ?$$

SKIP

What is the probability that exactly 3 2's turn up?

$$A = \{(2,2,2,1,1), (2,2,2,1,3), \dots$$
$$(2,1,2,4,2), \dots \}$$

$$|S| = 6 \cdot 6 \cdot 6 \cdot 6 \cdot 6 = 6^5$$

Multiplication principle Let $E_1, \ldots, E_m$ be m experiments. Suppose E_i has n_i ~~exper~~ outcomes. The "composite" experiment consisting of E_1 followed by E_2 followed by ... E_m has $n_1 \cdot n_2 \cdot \cdots n_m$ possible outcomes.

$\boxed{\text{Filling } r \text{ positions with } n \text{ objects}}$

Suppose you have n objects and you must use them to fill $r \leq n$ vacant spaces. How many ways can this be done?

Before answering this, we must specify whether

1. the order is important
2. the same object may be used multiple times.

Example $n = 5$ coins

P = penny
N = nickel
D = dime
Q = quarter
S = silver dollar

$r = 3$ positions. How many ways to fill the positions?

Case 1

Ordered, without replacement

- (P, N, D) and (P, D, N) are distinct outcomes, and
- (P, N, P) is not possible.

Case 2

Ordered, w/ replacement

- (P, N, D) and (P, D, N) are distinct
- (P, P, Q) and (P, P, P) are possible
- (P, P, Q) and (P, Q, P) are distinct

Case 3 : Unordered, without replacement

- No coin may be repeated
- (P, Q, S) and (S, P, Q) are the same outcome.

Case 4 : Unordered, w/ replacement

- ~~Th~~ coins may be repeated
- (P, P, Q) and (P, Q, P) are the same outcome.

Formulas

n objects, r samples

	w/ replacement	w/o replacement
ordered	n^r	$\frac{n!}{(n-r)!}$
unordered	$\frac{(n-1+r)!}{(n-1)!\,r!}$	$= \frac{n!}{r!\,(n-r)!}$

Derivations

All 4 formulas may be derived from the multiplication principle.

Ordered, w/ replacement

$1 \quad 2 \quad \cdots \quad n$

_ _ _ _ _ _ r spaces

Think of choosing the objects 1 at a time:

E_1: fill first space

E_2: " second "

Each experiment has n outcomes. By the multiplication principle, there are n^r total ways to fill the spaces.

Ordered, w/o replacement

E_1 has n outcomes,

E_2 has $n-1$ outcomes,

$\vdots$

Thus, by the mult. principle, there are

$$n(n-1)\cdot(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

ways. Each possible outcome is called a <u>permutation</u> of n objects taken r at a time. The notation

$$_nP_r = \frac{n!}{(n-r)!}$$

is sometimes used.

Example: How many <u>ordered</u> 5 card hands can be dealt from a deck of 52?

But, in card games, order doesn't matter (5! variations of each hand)

<u>Unordered, w/o replacement</u>

There are $r!$ ways to permute r objects.

~~Every~~

$$\left.\begin{array}{l}(P, N, Q)\\(P, Q, N)\\(N, P, Q)\\(N, Q, P)\\(Q, P, N)\\(Q, N, P)\end{array}\right\} 3! = 6 \text{ permutations}$$

These all count as the same.

Let ${}_nC_r$ denote the number of unordered samples w/ replacement. Then

$$({}_nC_r)\cdot(r!) = {}_nP_r$$

Therefore,

$${}_nC_r = \frac{{}_nP_r}{r!} = \frac{n!}{r!\,(n-r)!}$$

Each such unordered sample is called a <u>combination</u> of n objects taken r at a time.

Alternate notation:

$${}_nC_r = \binom{n}{r} \qquad \text{"n choose r"}$$

binomial coefficient.

<u>unordered, ~~[illegible]~~ with replacement</u>

See book (not so useful for probability)

Example Suppose a club has 10 people.
How many ways are there to select a president, vice-pres, secretary, & treasurer?

$n = 10, \quad r = 4$ ordered, w/o replacement

$$\Rightarrow \; {}_{10}P_4 = 10 \cdot 9 \cdot 8 \cdot 7 = \frac{10!}{6!} = 5040$$

Example How many possible 5-card poker hands can you be dealt from a 52 card deck?

$n = 52, \quad r = 5,$ unordered, w/o replacement

$$\Rightarrow \; {}_{52}C_5 = \binom{52}{5} = \frac{52!}{5!\,47!} = 2{,}598{,}960$$

How many ways are there to draw a spade flush (all spades)?

Exercise

$${}_{13}C_5 = \binom{13}{5} = 1287.$$

Thus, the prob. of being dealt a spade flush is

$$\frac{1287}{2598960} = 0.000495$$

<u>Example</u> 6-sided die, 5 rolls

$S = \{ \text{all outcomes} \}$

$A = \{ \text{exactly } 3 \text{ } 2\text{'s} \}$

$|S| = 6^5 = 7776$ (unordered, with replacement)

$|A| = \binom{5}{3} \cdot 5^2 = 20 \cdot 25 = 500$

$\Rightarrow P(A) = \dfrac{500}{6^5} = 0.064$

Distinguishable Permutations

What if some of the n objects are identical?

Simplest case: r objects of one type, $n-r$ of another.

<u>Example</u> 5 shapes

o o
△ △ △

There are $5!$ permutations, but not all are distinguishable:

o o △ △ △

o o △ △ △

The distinguishable permutations are

o o Δ Δ Δ
o Δ o Δ Δ
o Δ Δ o Δ
o Δ Δ Δ o
Δ o o Δ Δ
Δ o Δ o Δ
Δ o Δ Δ o
Δ Δ o o Δ
Δ Δ o Δ o
Δ Δ Δ o o

10 total.

Note $10 = \binom{5}{2} = {}_5C_2 =$ the number of ways of selecting the locations of the 2 o's.

In general, there are $\binom{n}{r}$ distinguishable permutations of n objects, r of one type, $n-r$ of another.

What if there are > 2 groups of non-distinguishable objects? The number of distinguishable permutations then is expressed in terms of <u>multinomial</u> coefficients (see book and HW)