Discrete distributions

<u>Definition</u> Let S_0 be the sample of some experiment.

- A <u>random variable</u> is a function X that assigns real numbers to elements of S_0.
- The <u>range</u> of X is the set $\{x \in \mathbb{R} :$ ~~Term~~ $X(s) = x$ for some $x \in S_0\}$.

<u>Remarks</u>

- Book uses "space" instead of "range"
- ~~"Support" is another term for "range".~~

A random variable (RV) effectively defines a new experiment whose sample space is the range of X, and which we denote S.

<u>Example</u> (Coin tossing)

$$S_0 = \{H, T\}$$

$$X(H) = 1$$

$$X(T) = 0$$

$$S = \{0, 1\}$$

Example (sum 2 dice)

$$S_0 = \{(i,j) : 1 \le i,j \le 6\}$$

$$X((i,j)) = i+j$$

$$S = \{2, 3, \dots, 12\}$$

Example Dr. Frankenstein creates a monster and wants to know its intelligence.

$$S_0 = \{\text{monsters}\}$$

$$X(s) = \text{IQ of } s \qquad (s = \text{a monster})$$

$$S = \{0, 1, 2, \dots, 300\}$$

Sometimes outcomes are already real numbers:

Example (roll 1 die)

$$S_0 = \{1, 2, 3, 4, 5, 6\}$$

$$X(i) = i$$

$$S = S_0$$

Purpose of RVs: They allow us to

- conceptualize outcomes
- simplify specification of probabilities
- use mathematical formulas to describe randomness
- compute "expected" outcomes

⋮

Definitions

- A set S is <u>countable</u> if it is finite or can be put in one-to-one correspondence w/ the positive integers.
- A random variable is <u>discrete</u> if its range is countable.

Assume X is discrete until further notice (Chap. 4)

Induced probability

If P_0 is a probability function on S_0,

then

$$P(X = x) = P_0(\{s : X(s) = x\})$$

is the <u>induced</u> probability function on S.

<u>Proposition</u> If P_0 satisfies the axioms, then so does P.

<u>Pf</u>: Exercise.

Often the notation

$$f(x) = P(X = x)$$

is used.

Q: What if P_0 is unknown (as is often the case)? How can we know that $f(x)$ defines a valid probability function?

A: Verify axioms directly.

__Def__ A __probability mass function__ of a discrete random variable X with range S is a function satisfying

(a) $f(x) > 0$ for all $x \in S$

(b) $\sum_{x \in S} f(x) = 1$

Then, P is defined via

$$P(X \in A) = \sum_{x \in A} f(x), \quad \text{for all } A \subset S.$$

__Remarks__
- Often ~~the~~ a p.m.f. is viewed as a function on all the reals by setting $f(x) = 0$ for $x \notin S$.
- The notations $P(A)$ and $P(X \in A)$ are interchangable.

<u>Terminology</u> The p.m.f. defines the "distribution" of X

<u>Example</u>: Discrete Uniform distribution

$$S = \{1, 2, \ldots, m\}$$

$$f(x) = \frac{1}{m}$$

<u>Example</u>: Not uniform: (sum of 2 dice)

$$S = \{2, 3, \ldots, 12\}$$

$f(2) = \frac{1}{36}$

$f(3) = \frac{2}{36}$

$f(4) = \frac{3}{36}$

$f(5) = \frac{4}{36}$

$f(6) = \frac{5}{36}$

$f(7) = \frac{6}{36}$

$f(8) = \frac{5}{36}$

$f(9) = \frac{4}{36}$

$f(10) = \frac{3}{36}$

$f(11) = \frac{2}{36}$

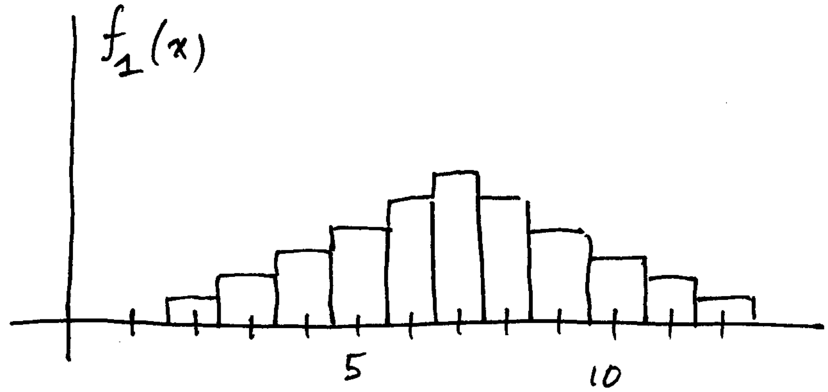
$f(12) = \frac{1}{36}$

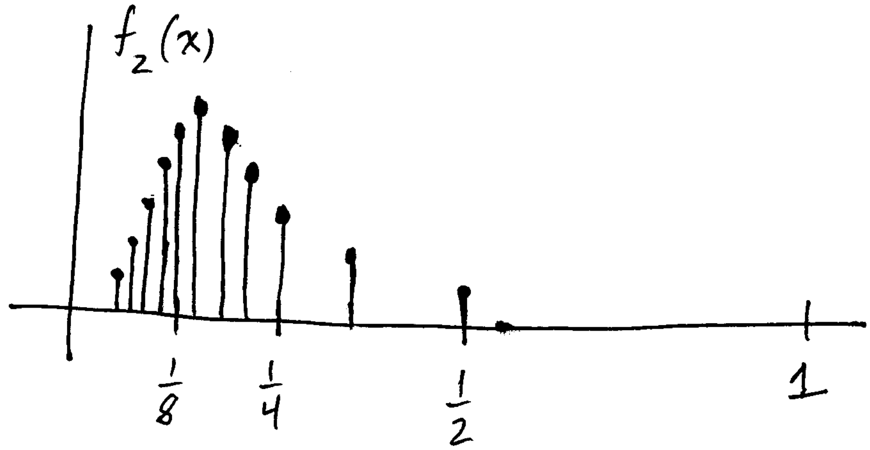
Visualizing discrete PMFs

- Probability histogram : when elements of S are equally spaced
- bar graph : otherwise

Ex X_1 = sum of 2 dice

$X_2 = 1/X_1$





Expectation

<u>Definition</u> Let X be a discrete random ~~value~~ variable with p.m.f $f(x)$ and range S. Let $u(X)$ be a function mapping S to $\mathbb{R}$. The expected value of $u(X)$ is defined by

$$E[u(X)] = \sum_{x \in S} u(x) f(x)$$

when the sum exists.

<u>Special cases</u>

1. When $u(X) = X$;

   $E[X]$ is called the <u>mean</u> of X.

2. when $u(X) = (X-\mu)^2$

   $E[(X-\mu)^2]$ is called the variance of X, denoted $Var(X)$.

<u>Notation</u> $\mu = E[X]$, $\sigma^2 = Var(X)$

The <u>standard deviation</u> is the square root of the variance, and denoted σ

<u>Example</u> Sum of 2 die $= X$

$$f(x) = \frac{6-|7-x|}{36}, \quad 2 \le x \le 12$$

$$E[X] = \sum_{x \in S} x f(x) = \sum_{x=2}^{12} x \cdot \left(\frac{6-|7-x|}{36} \right)$$

$$= 2 \cdot f(2) + 3 f(3) + \cdots + 12 f(12)$$

$$= 7$$

$$Var(X) = E[(X-7)^2] = \sum_{x=2}^{12} (x-7)^2 f(x)$$

$$= (2-7)^2 f(2) + (3-7)^2 f(3) + \cdots + (12-7)^2 f(12)$$

$$= \frac{210}{36} = 5.83$$

$$STD(X) = \sqrt{5.83} = 2.42$$

Suppose a friend offers you $1 if X is odd and will take $1 if X is even. What is your expected reward for playing this game?

$u(x) = 1$ if x is even

$u(x) = -1$ if x is odd

$$\text{expected payoff} = E[u(x)] = \sum_{x=2}^{12} u(x) f(x)$$

$$= f(2) - f(3) + f(4) - f(5) + \cdots + f(12)$$

$$= 0$$

Properties of Expectation

1) If a is a real number, then

$$E[a] = a.$$

2) ~~If $u_1(X)$, $u_2(X)$ are~~

$$E[u_1(X) + u_2(X)] = E[u_1(X)] + E[u_2(X)]$$

for any u_1, u_2

3) $$E[a\,u(X)] = a\,E[u(X)]$$ for any $a \in \mathbb{R}$, $u(X)$.

Proof ~~(1) $E[a] = \sum a$~~

(2) $$E[u_1(X) + u_2(X)] = \sum_{x \in S} (u_1(x) + u_2(x)) f(x)$$

$$= \sum_{x \in S} u_1(x) f(x) + \sum_{x \in S} u_2(x) f(x)$$

$$= E[u_1(X)] + E[u_2(X)]$$

(3) $$E[a\,u(X)] = \sum a\,u(x) f(x) = a \sum u(x) f(x)$$

$$= a\,E[u(X)].$$

(1) follows from (3) with $u(X) \equiv 1$.