Functions of Random Variables

If u is a function from $\mathbb{R}$ to $\mathbb{R}$, and $Y = u(X)$, where the dist of X is known, what is the dist. of Y?

Examples

- $u(x) = \frac{x-\mu}{\sigma}$

  $X \sim \cancel{N(0,1)}\ N(\mu, \sigma^2)$

  $Z = u(X) \implies Z \sim N(0,1)$

- $u(x) = \left(\frac{x-\mu}{\sigma}\right)^2, \quad X \sim N(\mu, \sigma^2)$

  $V = u(x) \implies V \sim \chi^2(1)$

Lognormal Dist

$$X \sim N(\mu, \sigma^2), \quad u(x) = e^x$$

$W = u(X) \implies W$ has a <u>lognormal dist</u>.

What is the pdf? Assume $w > 0$.

$$F(w) = P(W \le w) = P(e^X \le w)$$

$$= P(X \le \ln(w))$$

$$= \int_{-\infty}^{\ln(w)} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] dx$$

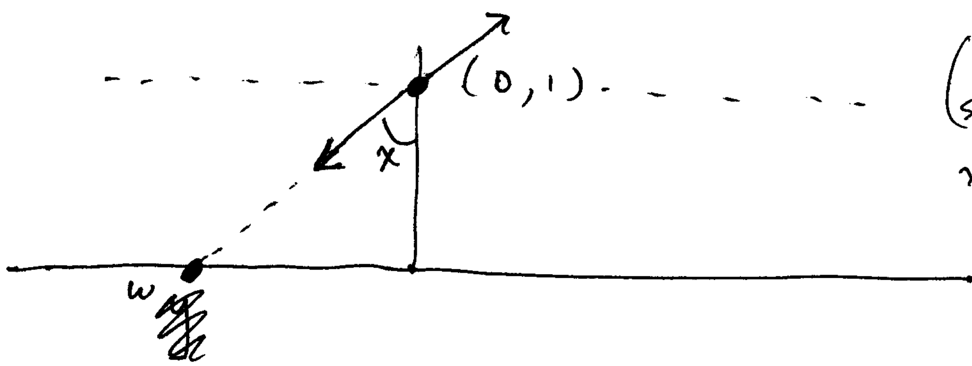
$$f(w) = F'(w) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(\overset{\ln w}{\cancel{x}}-\mu)^2}{2\sigma^2}\right] \cdot \left(\frac{1}{w}\right)$$

$$= \frac{1}{w\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(\ln w - \mu)^2}{2\sigma^2}\right], \quad w > 0.$$

Cauchy

$$X \sim \text{unif}\left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \quad u(x) = \tan x$$

$W = u(X)$ has a Cauchy dist.



(uniform spinner)

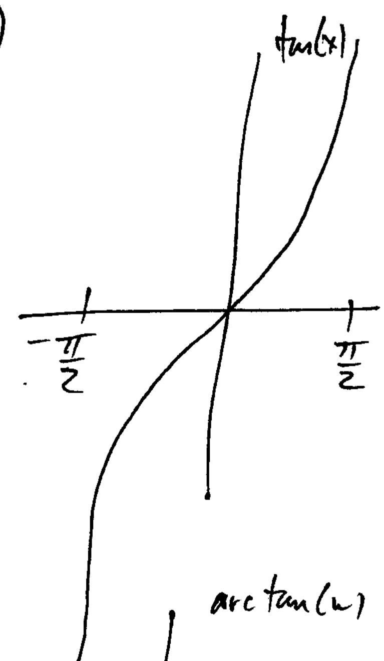
x = angle with y-axis

$$F(w) = P(W \le w) = P(\tan X \le w)$$

$$= P(X \le \arctan(w))$$

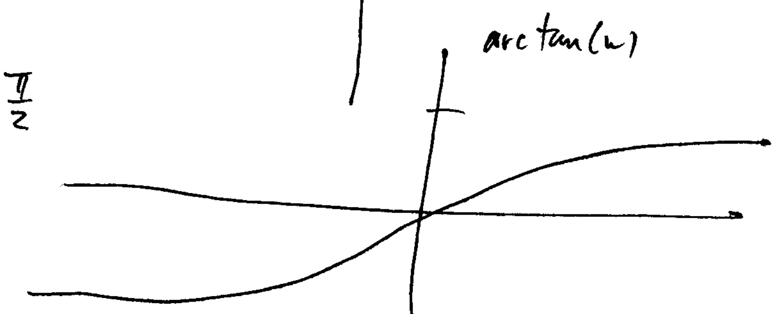
$$= G(\arctan(w)),$$

G = CDF of X



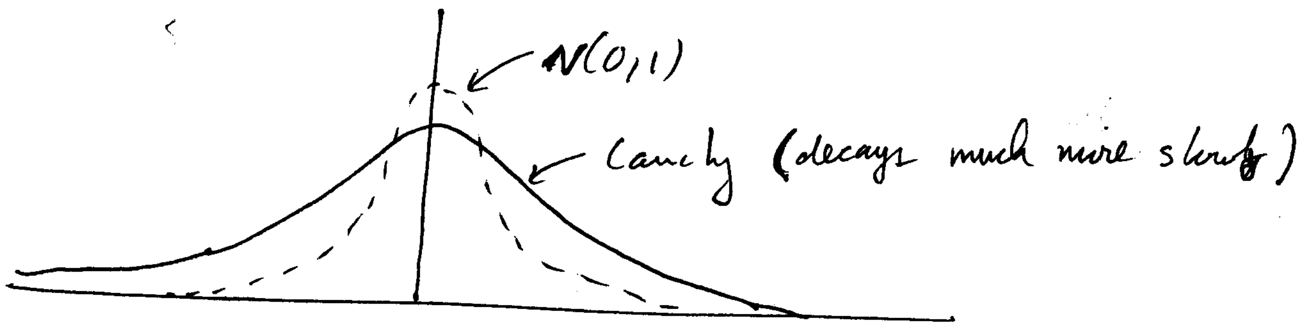
Recall

$$G(x) = \begin{cases} 0 & x < -\frac{\pi}{2} \\ \left(x + \frac{\pi}{2}\right)\frac{1}{\pi} & -\frac{\pi}{2} \le x \le \frac{\pi}{2} \\ 1 & x \ge \frac{\pi}{2} \end{cases}$$



$$f(w) = F'(w) = \frac{1}{\pi} \cdot \frac{\partial}{\partial w}(\arctan w)$$

$$= \frac{1}{\pi} \frac{1}{1 + w^2}$$



<u>Change of variable technique</u> : p 207 - 208

~~$X \sim f(x)$, $S_X = [c_1, c_2]$~~

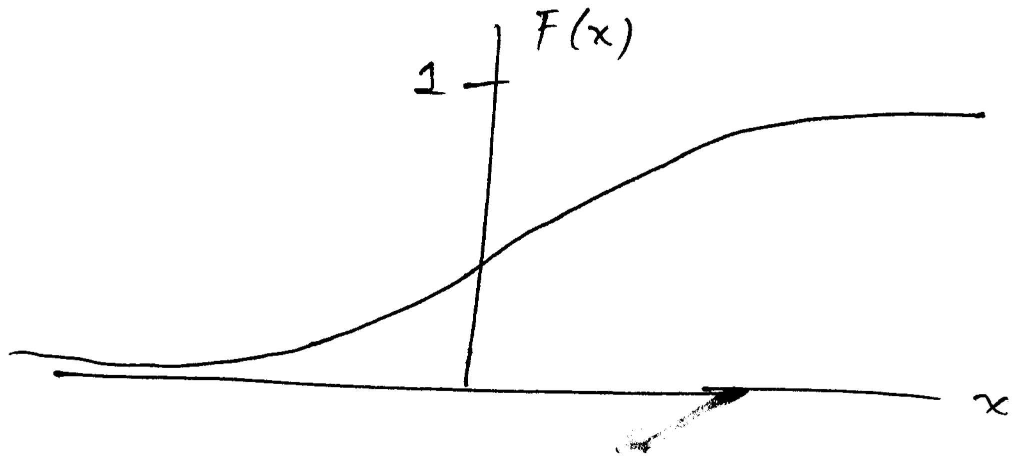
~~$Y = u(X) \sim g(y)$, $S_Y = [d_1, d_2]$~~

~~u increasing (so $d_1 = u(c_1)$, $d_2 = u(c_2)$)~~

~~<u>Ex</u> $f(x) = 3(1-x)^2$, $0 < x < 1$~~

~~$Y = (1-X)^3$~~

<u>Theorem</u> Let X be a continuous R.V. w/ CDF $F(x)$. If $Y = F(X)$, then $Y \sim \text{Unif}(0,1)$.



<u>Proof</u>: Clearly $S_Y = [0,1]$. Let $y \in [0,1]$. Let $G(y)$ denote the CDF of Y.

We must show $\boxed{G(y) = y}$ ← ask

$$G(y) = P(Y \leq y) = P(F(X) \leq y)$$
$$= P(X \leq F^{-1}(y)) = F(F^{-1}(y))$$
$$= y$$

∎

<u>Converse Theorem</u> Let $F : \mathbb{R} \to [0,1]$ be a continuous function such that

- $F(-\infty) = 0$
- $F(\infty) = 1$
- F is strictly increasing, i.e., if $x > y$ then $F(x) > F(y)$.

If $W \sim \text{Unif}(0,1)$ and $X = F^{-1}(W)$, then X is a continuous R.V. whose CDF is F.

(proof in book)

<u>Consequence</u>: We can simulate any continuous RV if we know its CDF and if we have a uniform random # generator.

Transformations of a discrete RV

Ex: Suppose $X \sim \text{poisson}(3)$

$Y = \sqrt{X}$. What is the pmf of Y?

$$X: f(x) = \frac{3^x \cdot e^{-3}}{x!}, \quad x = 0, 1, 2, \ldots$$

$$Y: g(y) = P(Y=y)$$

$$= P(\sqrt{X} = y)$$

$$= P(X = y^2) = f(y^2)$$

$$= \frac{3^{y^2} e^{-3}}{(y^2)!}, \quad y = 0, 1, \sqrt{2}, \sqrt{3}, \ldots$$