The Gamma Distribution

The *gamma distribution* has two real parameters, $\alpha > 0$ and $\theta > 0$, and pdf

$$f(x) = \begin{cases} \overbrace{\cancel{\frac{1}{\Gamma(\alpha)\theta^{\alpha}}}}^{C} \cdot x^{\alpha-1} e^{-x/\theta} & 0 \le x < \infty \\ 0 & \text{otherwise.} \end{cases}$$

where C is a constant such that $\int_0^\infty f(x)\,dx = 1$.

The constant C can be expressed in terms of the *Gamma function*:

$$\Gamma(t) = \int_0^\infty y^{t-1} e^{-y}\,dy \quad , \quad t > 0$$

In particular, $\frac{1}{C} = \int_0^\infty x^{\alpha-1} e^{-x/\theta}\,dx$ $[y = x/\theta]$ $dx = \theta\,dy$

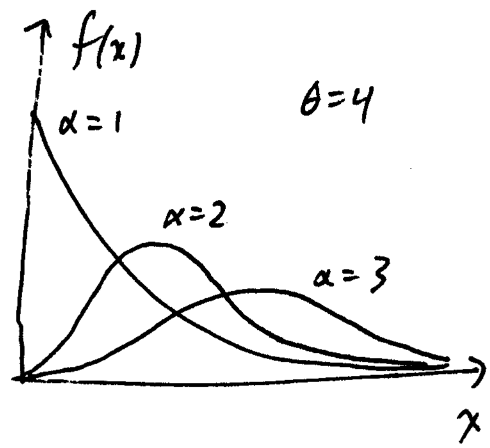
$$= \int_0^\infty \theta^{\alpha-1} \cdot y^{\alpha-1} e^{-y} \cdot \theta\,dy$$

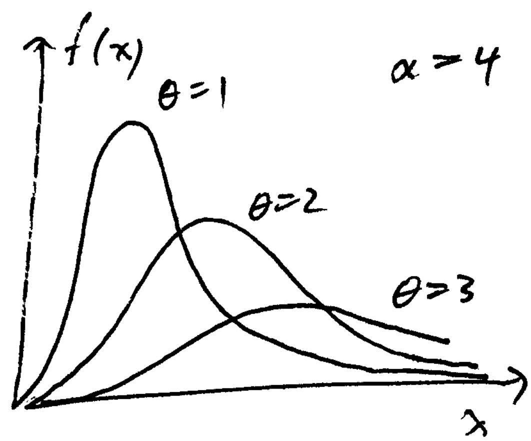
$$= \theta^{\alpha} \int_0^\infty y^{\alpha-1} e^{-y}\,dy = \theta^{\alpha}\,\Gamma(\alpha)$$

$$\Rightarrow \quad f(x) = \frac{1}{\Gamma(\alpha)\,\theta^{\alpha}}\, x^{\alpha-1} e^{-x/\theta} \quad , \quad x \geq 0$$

<u>Fact</u>: If n is a positive integer, then $\Gamma(n) = (n-1)!$ Thus Γ extrapolates the factorial function to all positive real numbers.

Effect of α, θ on pdf:





As α, θ increase, the mean and variance increase.

<u>HW</u> MGF: $M(t) = \frac{1}{(1-\theta t)^{\alpha}} \quad , \quad t < 1/\theta$

$$\mu = \alpha\theta \quad , \quad \sigma^2 = \alpha\theta^2$$

Special case: $\alpha = 1 \implies$

$$f(x) = \frac{1}{\Gamma(1)\,\theta} x^{1-1} e^{-x/\theta}$$

$$= \frac{1}{0!\,\theta} e^{-x/\theta} = \frac{1}{\theta} e^{-x/\theta}, \quad x \geq 0$$

$$\implies X \sim \exp(\theta)$$

Fact: Assume $X \sim \text{Poisson}(\lambda)$

Let α be a positive integer, and ~~then~~ let W be the ~~number of~~ waiting time until the α^{th} event in the Poisson process. Then

$$W \sim \text{Gamma}(\alpha, \theta),$$

where $\theta = 1/\lambda$. (see book for derivation)

Chi-square distribution

A gamma distribution with $\theta = 2$ and $\alpha = \frac{r}{2}$, where $r = 1, 2, 3, \dots$ is called a chi-square distrib. w/ r degrees of freedom.

Notation: $X \sim \chi^2(r)$

Pdf: $$f(x) = \frac{1}{\Gamma(r/2)\, 2^{r/2}} x^{\frac{r}{2}-1} e^{-x/2}$$

$$M(t) = \frac{1}{(1-2t)^{r/2}}, \quad t < \tfrac{1}{2}$$

$$\mu = \left(\frac{r}{2}\right)\cdot 2 = r, \quad \sigma^2 = \frac{r}{2}\cdot 2^2 = 2r.$$

The importance of the $\chi^2(r)$ distribution will become clear later in the course

$\chi^2(r)$ (upper) percentiles

Let $\alpha \in (0, \frac{1}{2})$ denote a probability.

Do not confuse w/ the first parameter of a Gamma distrib.

Assume $X \sim \chi^2(r)$. Define

$$\chi^2_\alpha(r) = \pi_{1-\alpha}$$

$$\chi^2_{1-\alpha}(r) = \pi_\alpha$$

That is, $\chi^2_\alpha(r)$ is a number such that

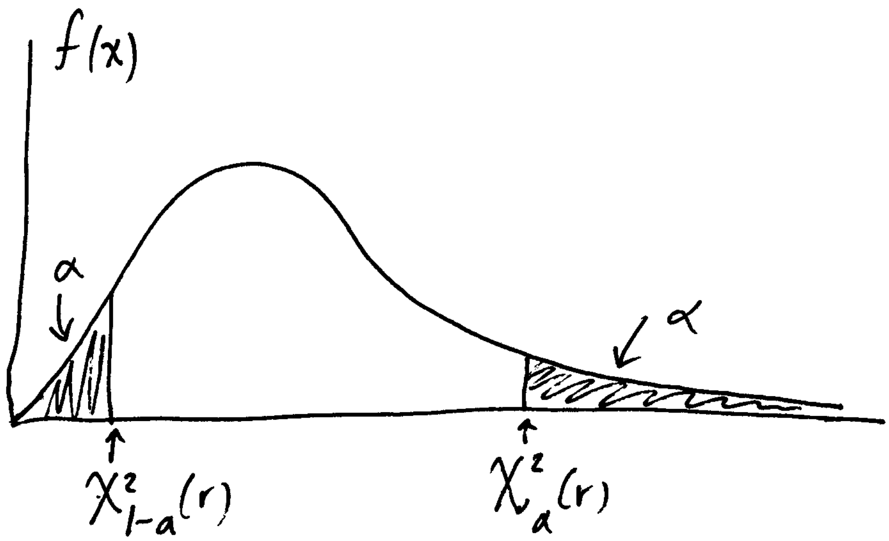
$$P\left(X \leq \chi^2_\alpha(r)\right) = 1-\alpha,$$

or equivalently,

$$P\left(X \geq \chi^2_\alpha(r)\right) = \alpha$$

Similarly,

$$P(X \le \chi^2_{1-\alpha}) = \alpha$$



Unfortunately, closed-form expressions for the CDF of the Gamma & $\chi^2(r)$ distributions. do not exist in general.

However, percentiles can be found using tables or computers.