$\boxed{\text{Independent Events}}$ (and Bayes Theorem).

Two events are independent if the occurance of one event does not change the probability of the other occurring.

Intuitively, we expect

$$P(A|B) = P(A)$$

and

$$P(B|A) = P(B)$$

when A & B are independent.

If this is the case, then by the multiplication rule,

$$P(A \cap B) = P(A) \cdot P(B|A) = P(A) \cdot P(B)$$

$$P(A \cap B) = P(B) \cdot P(A|B) = P(B) \cdot P(A).$$

<u>Definition</u> Two events A & B are <u>independent</u> iff $P(A \cap B) = P(A) \cdot P(B)$. Otherwise they are <u>dependent</u>.

<u>Remark</u>: If A & B are indep., and $P(B) > 0$, then

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A) \cdot P(B)}{P(B)} = P(A) \text{ as expected.}$$

<u>Example</u> Roll a fair 8-sided die.

What are two events?

Are they independent?

Iterate.

$$\left.\begin{array}{l} A = \text{odd} \\ B = \geq 5 \end{array}\right\} \text{indep.}$$

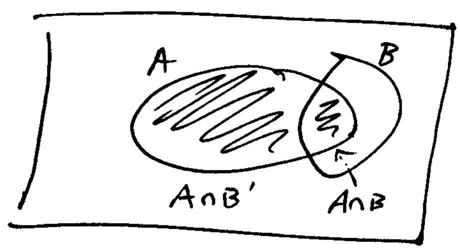
<u>Prop</u> If A & B are independent, then so are

(a) A & B'

(b) A' & B

(c) A' & B'

<u>Proof</u> (different from book) of (a):



$$A = (A \cap B) \cup (A \cap B')$$

$$P(A) = P(A \cap B) + P(A \cap B')$$

$$\begin{aligned} \Rightarrow P(A \cap B') &= P(A) - P(A \cap B) \\ &= P(A) - P(A) \cdot P(B) \\ &= P(A)\,(1 - P(B)) \\ &= P(A) \cdot P(B') \end{aligned}$$ $\blacksquare$

(b) & (c): exercises.

Independence of multiple events

<u>Definition</u>: Three events A, B & C are (<u>mutually</u>) <u>dependent</u> iff

(a) Each pair of events in indep.

(b) $P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$

More generally, for > 3 events to be indep., they must satisfy the multipication rule for each pair, triplet, quartet, etc.

<u>Example</u> Suppose a fair coin is tossed n times. What is the probability of at least 1 head turning up?

A_1 : 1st toss is heads

A_2 : 2nd " " "

$\vdots$

A_n

$B = A_1 \cup A_2 \cup \cdots \cup A_n$

Then

$$P(B) = 1 - P(B')$$

$$B' = (A_1 \cup A_2 \cup \cdots \cup A_n)'$$

$$= A_1' \cap A_2' \cap \cdots \cap A_n'$$

so

$$P(B') = P(A_1' \cap A_2' \cap \cdots \cap A_n')$$

$$= P(A_1') \cdot P(A_2') \cdots P(A_n') \quad \text{(by independence)}$$

$$= \frac{1}{2} \cdot \frac{1}{2} \cdot \cdots \frac{1}{2}$$

$$= \frac{1}{2^n}$$

$$\Longrightarrow P(B) = 1 - \frac{1}{2^n}$$

Note: this is somewhat easier than the direct way of enumerating all outcomes in B.

Example Draw two cards w/o replacement from a deck of 52

$A = \{\text{first card is a spade}\}$

$B = \{\text{second card is a diamond}\}$

$C = \{\text{second card is a K}\}$

- Are A & B independent?

  Intuitively, no. Formally,

$$P(A) = \frac{1}{4}, \quad P(B) = \frac{1}{4}$$

$$P(A \cap B) = P(A) \cdot P(B|A) = \frac{1}{4} \cdot \frac{13}{51} \neq \frac{1}{4} \cdot \frac{1}{4}$$

$$B = (B \cap A) \cup (B \cap A') \qquad \text{(disjoint)}$$

$$P(B) = P(B \cap A) + P(B \cap A')$$

$$= \frac{1}{4} \cdot \frac{13}{51} + P(A') \cdot P(B|A')$$

$$= \frac{1}{4} \cdot \frac{13}{51} + \frac{3}{4} \cdot$$

How to compute $P(B)$?

- A + C are indep.
- B + C are indep.
- Are A, B + C mutually indep? No, need all pairs to be.

Example: ~~what is~~ A fair die is rolled 5 times.
what is the prob. of getting exactly 3 2's?

Previously we say $P = \dfrac{\binom{5}{3} \cdot 5^2}{6^5} = .064 = \dfrac{500}{7776}$

Alternate derivation.

Consider the outcome

$$\underline{\;2 \mid \cancel{2} \mid 2 \mid 2 \mid \cancel{2}\;}$$

By independence, this event has probability

$$\frac{1}{6} \cdot \frac{5}{6} \cdot \frac{1}{6} \cdot \frac{1}{6} \cdot \frac{5}{6} = \left(\frac{1}{6}\right)^3 \cdot \left(\frac{5}{6}\right)^2$$

There are $\binom{5}{3}$ distinguishable permutations of the 2's, each with prob $\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^2$.

These $\binom{5}{3}$ events are disjoint so the overall event has prob

$$\binom{5}{3} \cdot \left(\frac{1}{6}\right)^3 \cdot \left(\frac{5}{6}\right)^2 = \frac{\binom{5}{3} 5^2}{6^5} = 0.064$$

as before.

Bayes' Theorem Let A & B be to events with $P(A) > 0$ & $P(B) > 0$.

Now

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

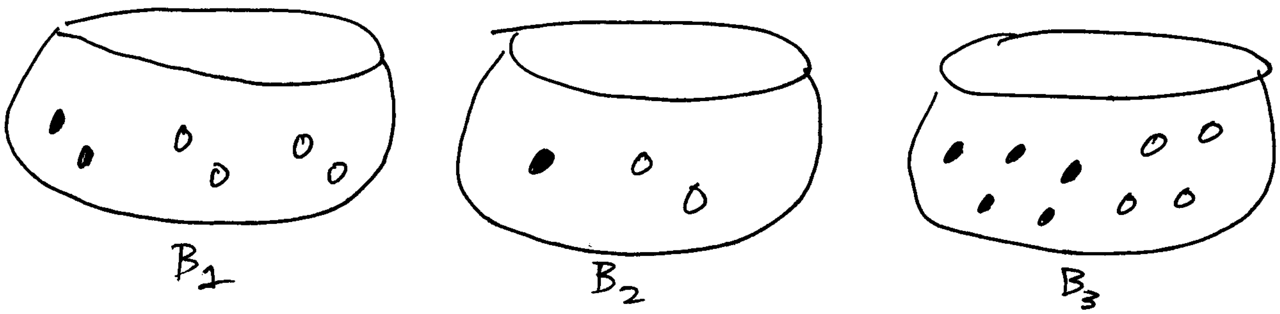
$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

$$\Longrightarrow \quad P(A) \cdot P(B|A) = P(A \cap B) = P(B)\,P(A|B)$$

$$\Longrightarrow \quad P(B|A) = \frac{P(B) \cdot P(A|B)}{P(A)}$$

Bayes' rule

<u>Example</u> ~~(book p. 103)~~



Experiment: choose bowl at random, then choose ball at random

Suppose a red chip is drawn. How likely are each of the bowls to have been selected?

R : red ball selected.

What are $P(B_1/R)$, $P(B_2/R)$ and $P(B_3/R)$?

$$P(B_1/R) = \frac{P(B_1)\cdot P(R/B_1)}{P(R)}$$

$$P(B_1) = \frac{1}{3}, \quad P(R/B_1) = \frac{2}{6}$$

<u>Side calculation</u>

$$P(R) = \;?$$

$$R = R \cap S = R \cap (B_1 \cup B_2 \cup B_3)$$

$$= (R \cap B_1) \cup (R \cap B_2) \cup (R \cap B_3)$$

(disjoint)

$$\Rightarrow P(R) = P(R \cap B_1) + P(R \cap B_2) + P(R \cap B_3)$$

$$= P(B_1)\cdot P(R/B_1) + P(B_2)\cdot P(R/B_2) + P(B_3) P(R/B_3)$$

$$= \frac{1}{3}\cdot\frac{2}{6} + \frac{1}{3}\cdot\frac{1}{3} + \frac{1}{3}\cdot\frac{5}{9} = \frac{3+3+5}{27}$$

$$= \frac{11}{27} =$$

So

$$P(B_1 \mid R) = \frac{\frac{1}{3} \cdot \frac{2}{6}}{\frac{11}{27}} = \frac{3}{11} = .273$$

Similarly,

$$P(B_2 \mid R) = \frac{P(B_2) \cdot P(R/B_2)}{P(R)} = \frac{\frac{1}{3} \cdot \frac{1}{3}}{\frac{11}{27}} = \frac{3}{11}$$

$$\Rightarrow P(B_3/R) = \frac{5}{11} = 0.455$$

More generally, if $B_1, \ldots, B_m$ form a partition of S, meaning

(a) $\bigcup_{i=1}^{m} B_i = S$

(b) $B_i \cap B_j = \phi$ if $i \neq j$

Then for any event A we have

$$A = (B_1 \cap A) \cup \cdots \cup (B_m \cap A)$$

and therefore

$$P(A) = \sum_{i=1}^{m} P(B_i \cap A)$$

$$= \sum_{i=1}^{m} P(B_i) \cdot P(A \mid B_i)$$

Combining this with Bayes' rule, we have

Bayes Theorem:

$$P(B_k \mid A) = \frac{P(B_k) \cdot P(A \mid B_k)}{\sum_{i=1}^{m} P(B_i) \cdot P(A \mid B_i)}$$

<u>Terminology</u>:

$P(B_k)$ = <u>prior</u> probability of B_k

$P(B_k|A)$ = <u>posterior</u> " " " given A