Multivariate Distributions

Discrete Bivariate Distributions

Let X, Y be two discrete RVs with ranges S_X, S_Y. Define $S = \{(x,y) : x \in S_x, y \in S_y\}$.

A function $f(x,y)$ is called the <u>joint pmf</u> of X & Y if

① $\sum_{(x,y) \in S} f(x,y) = 1$

② $0 \le f(x,y) \le 1$ for all x, y

③ $P\left((X,Y) \in A\right) = \sum_{(x,y) \in A} f(x,y)$

for all $A \subseteq S$.

Example Roll 2 dice.

X = smaller number

Y = larger number

What is $f(x,y)$?

$S_x = \{1, 2, \ldots, 6\}$

$S_y = \{1, 2, \ldots, 6\}$

y \ x	1	2	3	4	5	6
6	2/36	2/36	2/36	2/36	2/36	1/36
5	2/36	2/36	2/36	2/36	1/36	0
4	2/36	2/36	2/36	1/36	0	0
3	2/36	2/36	1/36	0	0	0
2	2/36	1/36	0	0	0	0
1	1/36	0	0	0	0	0

Def

1. Marginal pmf of X:

$$f_1(x) = \sum_{y \in S_Y} f(x,y)$$

2. Marginal pmf of Y

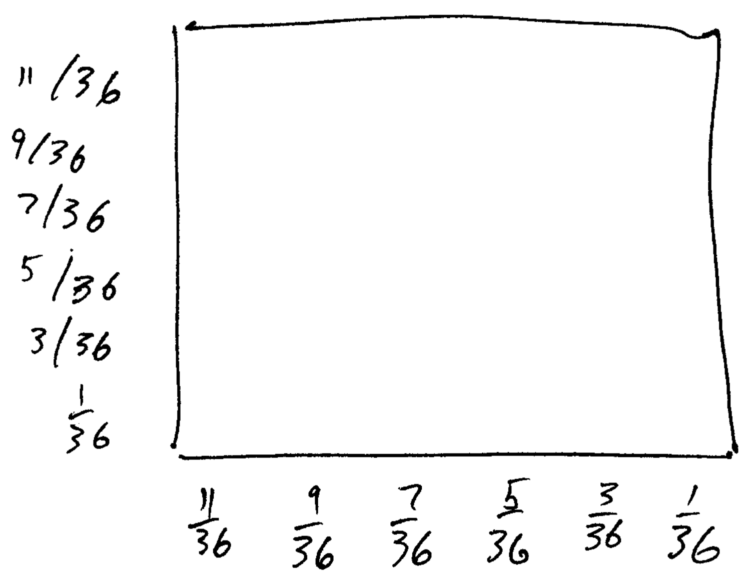
$$f_2(y) = \sum_{x \in S_X} f(x,y)$$

3. X and Y are independent iff

$$f(x,y) = f_1(x) f_2(y) \quad \text{for all } (x,y) \in S$$

Otherwise X, Y are dependent.

Example: Dice



Dependent:

$$f(1,1) \neq f_1(1) \cdot f_2(1)$$

$$\frac{1}{36} \qquad \frac{11}{36} \cdot \frac{1}{36}$$

Fact: $f_1(x) = P(X = x)$, $f_2(y) = P(Y = y)$.

Pf: Fix $\tilde{x} \in S_X$. Consider the event

$$A_{\tilde{x}} = \{(\tilde{x}, y) : \overset{x = \tilde{x}}{\cancel{y \in S_y}}\}.$$

Then $X = \tilde{x} \iff (X, Y) \in A_{\tilde{x}}$.

Thus, $P(X = \tilde{x}) = P((X, Y) \in A_{\tilde{x}})$

$$= \sum_{(x,y) \in A_{\tilde{x}}} f(x,y) = \sum_{y \in S_y} f(\tilde{x}, y) = f_1(\tilde{x}).$$

Similarly for f_2. $\square$

This justifies calling f_1, f_2 pmfs.

Expectation Assume X_1, X_2 are ~~joint~~ R.V.s w/ joint pmf $f(x_1, x_2)$.

If $u(x_1, x_2)$ is a function ~~from S~~ on $S = S_1 \times S_2$, then

$$E[u(X_1, X_2)] = \sum_{(x_1, x_2) \in S} u(x_1, x_2) f(x_1, x_2)$$

Can compute from marginals!

Mean of X_i: $u(x_1, x_2) = x_i$

$$\Longrightarrow E[u(X_1, X_2)] = E[X_i] = \mu_i$$

Variance of X_i: $u(x_1, x_2) = (x_i - \mu_i)^2$

$$\Longrightarrow E[u(X_1, X_2)] = E[(X_i - \mu_i)^2]$$

$$= \sigma_i^2 = Var(X_i).$$

Covariance of X_1, X_2 $u(x_1, x_2) = (X_1 - \mu_1) \cdot (X_2 - \mu_2)$

$$\Longrightarrow E[u(X_1, X_2)] = E[(X_1 - \mu_1)(X_2 - \mu_2)]$$

$$= Cov(X_1, X_2)$$

Bivariate Continuous RVs

Assume X, Y are continuous RVs.

The Joint PDF of (X, Y) satisfies

(1) $f(x,y) \geq 0$

(2) $\int_S f(x,y)\,dx\,dy = 1$

(3) $P\big((X,Y) \in A\big) = \int_A f(x,y)\,dx\,dy$ for all $A \subseteq S$.

volume of region under surface $f(x,y)$

Marginals:

$$f_1(x) = \int_{-\infty}^{\infty} f(x,y)\,dy \quad , \quad x \in S_1$$

$$f_2(y) = \int_{-\infty}^{\infty} f(x,y)\,dx \quad , \quad y \in S_2$$

Independence: $\Leftrightarrow f(x,y) = f_1(x)\, f_2(y)$.

Expectation: $E[u(X,Y)] = \int_{(x,y)\in S} u(x,y)\, f(x,y)\,dx\,dy$.

Fact If $f(x,y)$ is a joint pmf / pdf, then $f_1(x)$, $f_2(y)$ are pmfs / pdfs in their own right.

Proof (discrete case).

(a) $f_1(x) = \sum_{y \in S_2} f(x,y)$

is the sum of non-negative terms, so it is non-negative.

(b) $\sum_{x \in S_1} f_1(x) = \sum_{x \in S_1} \sum_{y \in S_2} f(x,y)$

$= \sum_{(x,y) \in S} f(x,y) = 1$

by def of joint pmf. ∎

Thus, f_1, f_2 define distributions on X, Y.

Moreover, f_1, f_2 are the pmfs / pdfs of X, Y when viewed as univariate random variables.

(c) ~~deffered~~ deferred.

Example 2 dice
X = smaller
Y = larger

Ignore Y. What is pmf of X?

$$P(X = 6) = \frac{1}{36} \qquad (6,6)$$
$$P(X = 5) = \frac{3}{36} \qquad (5,5),\ (5,6),\ (6,5)$$
$$\vdots \qquad \frac{5}{36}$$
$$\frac{7}{36}$$
$$\frac{9}{36}$$
$$\frac{11}{36}$$

$\Longrightarrow$ same as marginal $f_1(x)$ we derived earlier!

$\boxed{\text{Proof of (c)}}$ Must show for each $\tilde{x} \in S_1$,
$$P_1(X = \tilde{x}) = f_1(\tilde{x}).$$

Define $A_{\tilde{x}} = \{ (x,y) \in S \ : \ x = \tilde{x} \}$

Observe $X = \tilde{x} \iff (X,Y) \in A_{\tilde{x}}$.

Thus, $P_1(X=\tilde{x}) = P(A_{\tilde{x}}) = \displaystyle\sum_{(x,y)\in A_{\tilde{x}}} f(x,y)$

$$= \sum_{y \in S_2} f(\tilde{x},y) = f_1(\tilde{x}) \qquad \blacksquare$$

<u>Another example</u> Deck of 13 cards (spades)

X = first card dealt (w/o replacement)

Y = second " "

	A	2	3									Q	K	
A	0													$\frac{1}{13}$
2	$\frac{1}{13}\cdot\frac{1}{12}$	0												$\frac{1}{13}$
3	$\frac{1}{13}\cdot\frac{1}{12}$		0											$\frac{1}{13}$
	⋮			0										⋮
					0									
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							0							
								0						
									0					
										0				
											0			
Q												0		
K													0	
	$\frac{1}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	...										

$$f(x,y) = P(X=x, Y=y) = P(X=x)\cdot P(Y=y \mid X=x)$$

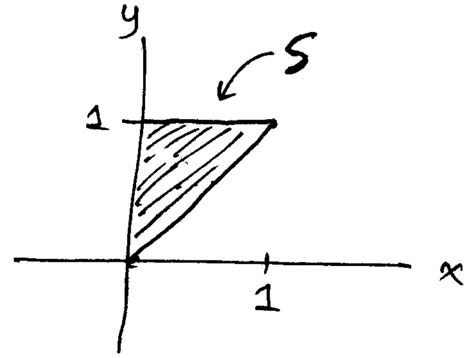
$$= \frac{1}{13}\cdot\frac{1}{12} \quad \text{(all non-diagonal squares)}$$

$$f_1(x) = \frac{1}{13} = f_2(y) \qquad [\text{uniform!}]$$

Example $\quad f(x,y) = 2, \quad 0 \le x \le y \le 1$

$$f_1(x) = \int_x^1 2\,dy = 2(1-x)$$

$$f_2(y) = \int_0^y 2\,dx = 2y$$



Exercise: (1) Find marginals

(2) Are X, Y indep.?

(3) What is $P(\{(x,y) : y \ge \frac{1}{2}\})$?

(4) What is $P(\{(x,y) : x \ge \frac{1}{2} \text{ and } y \ge \frac{1}{2}\})$?

Sol (1) above

(2) No

(3) $\cancel{\frac{1}{4}}$, $\frac{3}{4}$ by volume considerations, or

$$\int_{\frac{1}{2}}^1 2y\,dy = y^2\Big|_{1/2}^1 = 1 - \frac{1}{4} = \frac{3}{4}$$

(4) $\frac{1}{4}$ by volume considerations, or

$$\int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 2\,dy\,dx = \int_{\frac{1}{2}}^1 \int_x^1 2\,dy\,dx = \int_{\frac{1}{2}}^1 (2-2x)\,dx$$

$$= 1 - \left[x^2\right]_{\frac{1}{2}}^1 = \frac{1}{4} \checkmark$$

(8)

Bivariate Normal

Parameters : $\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \rho$ $(-1 \le \rho \le 1)$

$$f(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left(-\frac{q(x,y)}{2}\right) \quad \text{where}$$

$$q(x,y) = \frac{1}{1-\rho^2}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]$$

Facts

$E[X] = \mu_X$

$E[Y] = \mu_Y$

$Var[X] = \sigma_X^2$

$Var[Y] = \sigma_Y^2$

$Cov(X,Y) = \rho\sigma_X\sigma_Y$

X & Y are independent $\iff \rho = 0$.

Skim the rest of Ch 5.

<u>Fact</u>: If X_1, X_2 are indep, and $u(x_1, x_2) = u_1(x_1)\, u_2(x_2)$, then $E[u_1(X_1) u_2(X_2)]$

$$= E[u_1(X)] \cdot E[u_2(X_2)].$$

<u>Pf</u> Assume $\mathbf{X}_1, \mathbf{X}_2$ are discrete (continuous case: exercise)

Then
$$E[u(X_1, X_2)] = \sum_{(x_1, x_2) \in S} u_1(x_1) u_2(x_2) f(x_1, x_2)$$

$$= \sum_{(x_1, x_2) \in S} u_1(x_1) u_2(x_2) f_1(x_1) f_2(x_2)$$

$$= \left(\sum_{x_1 \in S_1} u_1(x_1) f_1(x_1) \right) \left(\sum_{x_2 \in S_2} u_2(x_2) f_2(x_2) \right)$$

$$= \sum_{x_1} \left[u_1(x_1) f_1(x_1) \sum_{x_2} u_2(x_2) f_2(x_2) \right]$$

$$= E[u_1(X_1)]\, E[u_2(X_2)] \qquad \blacksquare$$

<u>Corollary</u> If X, Y are indep, then $\operatorname{Cov}(X, Y) = 0$.

~~Define correlation~~

__Definition__ The __correlation coefficient__ for (X, Y) is defined as $\rho = \dfrac{Cov(X,Y)}{\sigma_X \sigma_Y}$.

X & Y are said to be __uncorrelated__ iff $\rho = 0$. Otherwise, they are __correlated__.

__Facts__

① $|\rho| \le 1$

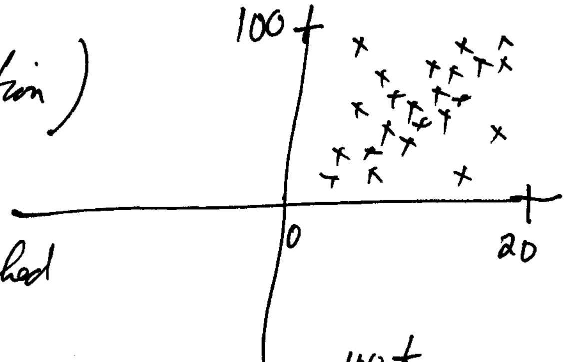
② X, Y independent $\Longrightarrow \rho = 0$

③ $Y = X \rightarrow \rho = 1$; $Y = -X \rightarrow \rho = -1$

$\boxed{\text{Example}}$ ① X = hours spent studying for exam

Y = grade on exam

$\rho > 0$ (positive correlation)



② X = ~~hours of sleep night before taking exam~~ hours of TV watched week before taking exam

Y = exam grade

$\rho < 0$ (negative correlation)