Functions of a Normal Random Sample

Assume $X_i \sim N(\mu, \sigma^2)$, $i = 1, \dots, n$

and X_i are independent.

We will show that

① $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \sim N\left(\mu, \frac{\sigma^2}{n}\right)$

② $\sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma}\right)^2 \sim \chi^2_{(n)}$

③ $\frac{(n-1)S^2}{\sigma^2} = \frac{\sum (X_i - \overline{X})^2}{\sigma^2} \sim \chi^2(n-1)$

$\boxed{\text{Prop}}$ $\bar{X} \sim N(\mu, \sigma^2/n)$.

Proof: Let $M(t) = \exp\left\{\mu t + \frac{\sigma^2 t^2}{2}\right\}$ be the MGF of X_i. By an earlier corollary,

$$M_{\bar{X}}(t) = \left[M\left(\frac{t}{n}\right)\right]^n$$

$$= \left[\exp\left(\mu\left(\frac{t}{n}\right) + \frac{\sigma^2 (t/n)^2}{2}\right)\right]^n$$

$$= \exp\left(\mu t + \frac{(\sigma^2/n)t^2}{2}\right)$$

This is the MGF of $N(\mu, \sigma^2/n)$.

Since the MGF uniquely determines a distribution, we conclude $\bar{X} \sim N(\mu, \sigma^2/n)$.

Prop Assume

$$X_1 \sim \chi^2(r_1)$$
$$X_2 \sim \chi^2(r_2)$$
$$\vdots$$
$$X_n \sim \chi^2(r_n)$$

and $X_1, \dots, X_n$ are indep. If

$$Y = X_1 + X_2 + \dots + X_n, \text{ then}$$

$$Y \sim \chi^2(r_1 + r_2 + \dots + r_n).$$

Proof. Recall the MGF of $\chi^2(r)$:

$$M(t) = \frac{1}{(1-2t)^{r/2}}, \quad t < \frac{1}{2}$$

Thus
$$M_Y(t) = \prod_{i=1}^{n} M_i(t) =$$
$$\prod_{i=1}^{n} (1-2t)^{-\frac{r_i}{2}}$$
$$= (1-2t)^{-\frac{(r_1+r_2+\dots+r_n)}{2}}$$

$= \text{MGF of } \chi^2(r_1 + r_2 + \dots + r_n)$ ∎

Corollary.

Recall that if $Z \sim N(0,1)$, then

$Z^2 \sim \chi^2(1)$.

Also recall that if $X \sim N(\mu, \sigma^2)$,

then $\frac{X-\mu}{\sigma} \sim N(0,1)$ and

$\left(\frac{X-\mu}{\sigma}\right)^2 \sim \chi^2(1)$.

Therefore, $\sum \frac{(X_i-\mu)^2}{\sigma^2}$

$$= \underbrace{\sum_{i=1}^{n} \left(\frac{X_i-\mu}{\sigma}\right)^2}_{\text{Sum of } n \text{ independent } \chi^2(1) \text{ RVs.}} \sim \chi^2(n)$$

<u>Prop</u> If $X_1, \dots, X_n \sim N(\mu, \sigma^2)$ are indep., then

(a) $\bar{X}$ and S^2 are indep.

(b)
$$\frac{(n-1)S^2}{\sigma^2} = \sum_{i=1}^{n} \left(\frac{X_i - \bar{X}}{\sigma} \right)^2 \sim \chi^2(n-1)$$

<u>Proof</u> : See book.

<u>Exercise</u> : Assume $X_i \sim N(\mu_i, \sigma_i^2)$, $i = 1, \dots, n$ are indep. Let $Y = \sum a_i X_i$.

What is the dist. of Y?

<u>Sol</u>: $N(\sum a_i \mu_i, \sum a_i^2 \sigma^2)$

(see book p. 304)