Ordered vs Unordered

We previously considered the following example.

Example Suppose a fair, 6-sided die is rolled 5 times. What is the probability of rolling exactly 3 2s?

The solution I presented went like this:

$$S = \{ (x_1, \dots, x_5) : x_i = 1,2,3,4,5, \text{ or } 6 \}$$

$$A = \{ (x_1, \dots, x_5) \in S : 3\ x_i\text{s are } 2 \text{ and the other } 2\ x_i\text{s are not } 2 \}.$$

Then $P(A) = \dfrac{N(A)}{N(S)}$.

In computing $N(A)$ and $N(S)$, the question arose of whether the outcomes of the experiment should be viewed as ordered or unordered 5-tuples. One could argue that we only care that there are 3 2's, not what order they come in, and therefore $(2,2,3,4,2)$ and $(3,2,2,4,2)$ are really the same outcome. After all, this is how we treat poker hands.

The solution to this conundrum has multiple parts:

- If we are going to use the formula
$$P(A) = \frac{N(A)}{N(S)},$$
we must view the rolls as <u>ordered</u>, because the formula assumes $\boxed{\text{equally likely outcomes}}$. The outcomes are only equally likely when the rolls are ordered.

- It is not wrong to view the outcomes as unordered 5-tuples, but then the outcomes are not equally likely, and we cannot apply the formula $P(A) = N(A)/N(S)$.

- Regarding the question of cards dealt from a deck: It is actually valid to view the outcomes as either ordered <u>or</u> unordered. The outcomes are equally likely either way. This is because the 52 cards are distinct and we are sampling without replacement. Viewing hands as ordered, the probability of a spade flush is

$$\frac{{}_{13}P_5}{{}_{52}P_5} = \frac{\frac{13!}{7!}}{\frac{52!}{47!}} = \frac{\frac{13!}{7!\,5!}}{\frac{52!}{47!\,5!}} = \frac{{}_{13}C_5}{{}_{52}C_5},$$

which is the answer you get when not considering order to be important.

To drive the point home lets consider a very simple example:

<u>Example</u> A fair coin is tossed twice in succession. What is the probability of observing 2 heads?

Clearly the answer should be $\frac{1}{4}$.

Viewing the ~~~~ tosses as ordered:

$$S = \{HH, HT, TH, TT\}$$

$$A = \{HH\}.$$

These outcomes are equally likely, so $P(A) = \frac{N(A)}{N(S)} = \frac{1}{4}$ as expected. Now, if we view the ~~events~~ tosses as unordered, we have

$$S = \{HH, HT = TH, TT\}$$

$$A = \{HH\}.$$

If we apply the formula $P(A) = \frac{N(A)}{N(S)}$ we get $P(A) = 1/3$. However, it is not valid to apply the formula ~~~~ because the outcomes are <u>not</u> equally likely. Indeed, $P(HH) = \frac{1}{4}$, $P(HT) = \frac{1}{2}$, $P(TT) = \frac{1}{4}$. We are twice as likely to get a head and a tail as we are to get to heads. So there is no contradiction.

Returning to the dice example:

Clearly $(2,2,2,2,2)$ and $(1,2,3,4,5)$ are not equally likely, if we disregard order. There are many ordered 5-tuples corresponding to the second, but only 1 corresponding to the first.

The ordered 5-tuples are equally likely, so we may base our computation on them. We have

$$N(S) = 6^5$$

$$N(A) = \binom{5}{3} \cdot 5^2$$

because there are $\binom{5}{3}$ ways to position the 3 2s and 5×5 ways to fill the other 2 positions with non-2s.

Thus $$P(A) = \frac{\binom{5}{3} 5^2}{6^5} = \frac{10 \cdot 25}{7776} = 0.032.$$

More generally, this discussion sheds light on why we don't usually consider sampling w/replacement together with unordered pairs: because the outcomes are not equally likely and we can't use counting methods to directly compute the probability.