Hypothesis Testing

Suppose $X \sim f(x;\theta)$, $\theta \in \Theta$ is unknown.

Also suppose $\Theta = \Theta_0 \cup \Theta_1$, where

$\Theta_0 \cap \Theta_1 = \phi$.

Θ_0 and Θ_1 correspond to hypotheses about X:

$$H_0 : \theta \in \Theta_0$$
$$H_1 : \theta \in \Theta_1$$

A <u>test</u> of H_0 vs. H_1 is a rule that decides which of Θ_0 or Θ_1 is true given a random sample $X_1, \ldots, X_n$

Example $X \sim \text{binom}(1, p)$, p unknown
(Bernoulli trial)

$\Theta = [0, 1]$

e.g., we have a biased coin with unknown p.

Suppose we want to decide whether the coin favors heads or tails.

$$\Theta_0 = [0, \tfrac{1}{2})$$
$$\Theta_1 = [\tfrac{1}{2}, 1]$$

In other words, we have the two hypotheses

$$H_0 : \ p < \tfrac{1}{2}$$
$$H_1 : \ p \geq \tfrac{1}{2}.$$

If $X_1, \dots, X_n$ is a random sample,

then a reasonable test is

decide H_0 is true if $\frac{1}{n} \sum X_i < \frac{1}{2}$

" H_1 " " " " $\geq \frac{1}{2}$

~~Another~~ Often H_0 describes a state of affairs where things are normal (as in "not unusual," not the normal distribution) are unchanged, while H_1 reflects an abnormal state.

Ex ~~H~~ In the previous example, consider

$$H_0 : p = \frac{1}{2}$$

$$H_1 : p \neq \frac{1}{2}$$

For this reason, the following terminology is used:

H_0 is called the <u>null hypothesis</u>

H_1 " " " <u>alternative hypothesis</u>

More <u>terminology</u>

<u>Composite</u> Hypothesis : H_i has > 1 element (e.g. $p \neq \frac{1}{2}$)

<u>Simple</u> " H_i has 1 " (e.g. $p = \frac{1}{2}$)

<u>Type I error</u> : declare H_1 true | H_0 true

<u>Type II error</u> " H_0 " | H_1 "

We will study tests of

$$H_0 : \mu = \mu_0$$ simple

versus

$$H_1 : \mu \neq \mu_0 \quad \text{(two-sided)}$$

or $$H_1 : \mu > \mu_0 \quad \text{(one-sided)}$$

or $$H_1 : \mu < \mu_0 \quad \text{(one-sided)}$$

composite

where $X \sim N(\mu, \sigma^2)$, μ unknown, σ^2 known or unknown.

Assume $X_1, \dots, X_n \sim N(\mu, \sigma^2)$, σ^2 known

Consider

$$H_0 : \mu = \mu_0$$
$$H_1 : \mu \neq \mu_0$$

where μ_0 is fixed and known

(Example: X is the weight of bags of flour produced by a company. The typical weight is μ_0, but they suspect the process may be off slightly)

What is a reasonable test? Declare H_1 when

$$|\bar{X} - \mu_0| > \sigma/\sqrt{n}$$

$$|\bar{X} - \mu_0| > 2\sigma/\sqrt{n}$$

$$\vdots$$

$$|\bar{X} - \mu_0| > \lambda\sigma/\sqrt{n}, \quad \lambda > 0$$

How many standard deviations (of $\bar{X}$) is $\bar{X}$ from μ_0?

How do we decide which ~~cutoff~~ cutoff (λ) to use?

What if we could ~~control~~ set λ such that the probability of a type I error was equal to some pre-specified level of significance α,

i.e.

$$P(\text{declare } H_1 \mid H_0 \text{ true}) = \alpha.$$

For our problem, that means choosing $\lambda = \lambda_\alpha$ such that

$$P\left(\left|\frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}\right| > \lambda_\alpha \,\middle|\, H_0 \text{ true}\right) = \alpha$$

Recall, H_0 true $\iff$ $\mu = \mu_0$, so if H_0 is true then

$$\bar{X} \sim N\left(\mu_0, \frac{\sigma^2}{n}\right)$$

and hence

$$\frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \sim N(0,1)$$

Thus, we want λ_α such that

$$P(|Z| > \lambda_\alpha) = \alpha$$

$$\parallel$$

$$P(-\lambda_\alpha < Z < \lambda_\alpha)$$

where $Z \sim N(0,1)$.

$$\Longrightarrow \lambda_\alpha = z_{\alpha/2}$$

Thus, in order to have a test whose level of significance is α, we should

$$\text{declare } H_1 \iff \frac{|\bar{X}-\mu_0|}{\sigma/\sqrt{n}} > z_{\alpha/2}$$

$$\iff |\bar{X}-\mu_0| > z_{\alpha/2}\left(\frac{\sigma}{\sqrt{n}}\right)$$

$$\iff \mu_0 \in \left(\bar{X} - z_{\alpha/2}\left(\frac{\sigma}{\sqrt{n}}\right), \bar{X} + z_{\alpha/2}\left(\frac{\sigma}{\sqrt{n}}\right)\right)$$

Thus, ~~the so-called~~ we accept H_0 iff μ_0 is in the $100(1-\alpha)\%$ conf. interval for μ_0.

Think about this: could we repeat the above procedure if we instead wanted to control the type II probability of error?

$\boxed{\text{Exercise}}$ (in class)

Consider testing

$$H_0 : \mu = \mu_0 \quad \text{vs.} \quad H_1 : \mu > \mu_0$$

with

$$\frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda_\alpha .$$

What should λ_α be to ensure $P(\text{type I error}) = \alpha$?

Solution

$$P(\text{type I error}) = P(\text{declare } H_1 \mid H_0 \text{ true})$$

$$= P\left(\frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} > \lambda_\alpha \,\middle|\, H_0 \text{ true} \right)$$

$$= P(Z > \lambda_\alpha) = \alpha$$

$$\Rightarrow \lambda_\alpha = z_\alpha$$

Exercise $\quad H_1 = \mu < \mu_0$

$$\Rightarrow \lambda_\alpha = -z_\alpha$$

$\boxed{\text{Unknown Variance}}$ What if σ^2 is unknown?

Idea: replace $\dfrac{\bar{X}-\mu_0}{\sigma/\sqrt{n}}$ by $T = \dfrac{\bar{X}-\mu_0}{S/\sqrt{n}}$

where $$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

and $$S = \sqrt{S^2}.$$

~~Then a test is~~ Recall $T \sim t(n-1)$

Therefore, the test

$$\begin{cases} H_1 & \text{if } |T| > t_{\alpha/2}(n-1) \\ H_0 & \text{if } |T| < t_{\alpha/2}(n-1) \end{cases}$$

Has a level of significance (probability of Type I error) equal to α.