Stat310 HW5 Solutions (100')

(5')

3.5–2 Here $x = \sqrt{y}$, $D_y(x) = 1/2\sqrt{y}$ and $0 < x < \infty$ maps onto $0 < y < \infty$. Thus

$$g(y) = \sqrt{y}\left|\frac{1}{2\sqrt{y}}\right| = \frac{1}{2}e^{-y/2}, \qquad 0 < y < \infty.$$

(10')

3.5–6 It is easier to note that

$$\frac{dy}{dx} = \frac{e^{-x}}{(1+e^{-x})^2} \qquad \text{and} \qquad \frac{dx}{dy} = \frac{(1+e^{-x})^2}{e^{-x}}.$$

Say the solution of x in terms of y is given by x^*. Then the p.d.f. of Y is

$$g(y) = \frac{e^{-x^*}}{(1+e^{-x^*})^2}\left|\frac{(1+e^{-x^*})^2}{e^{-x^*}}\right| = 1, \qquad 0 < y < 1,$$

as $-\infty < x < \infty$ maps onto $0 < y < 1$. Thus Y is $U(0,1)$.

(10')

3.5–12

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} \frac{x}{\pi(1+x^2)}\,dx \\ &= \lim_{a\to-\infty}\left[\frac{1}{2\pi}\ln(1+x^2)\right]_a^0 + \lim_{b\to+\infty}\left[\frac{1}{2\pi}\ln(1+x^2)\right]_0^b \\ &= \frac{1}{2\pi}\left[\lim_{a\to-\infty}\{-\ln(1+a^2)\} + \lim_{b\to+\infty}\ln(1+b^2)\right]. \end{aligned}$$

(5')

4.1–4 $\dfrac{25!}{7!8!6!4!}(0.30)^7(0.40)^8(0.20)^6(0.10)^4 = 0.00405.$

(10')

4.1–8 (a)

$$\begin{aligned} P\left(0 \le X \le \tfrac{1}{2}\right) &= \int_0^{\frac{1}{2}}\int_{x^2}^1 \frac{3}{2}\,dy\,dx \\ &= \int_0^{\frac{1}{2}} \frac{3}{2}(1-x^2)\,dx = \frac{11}{16}; \end{aligned}$$

(b)

$$\begin{aligned} P\left(\tfrac{1}{2} \le Y \le 1\right) &= \int_{\frac{1}{2}}^1\int_0^{\sqrt{y}} \frac{3}{2}\,dx\,dy \\ &= \int_{\frac{1}{2}}^1 \frac{3}{2}\sqrt{y}\,dy = 1 - \left(\frac{1}{2}\right)^{3/2}; \end{aligned}$$

(c) $$\begin{aligned} P\left(\tfrac{1}{2} \le X \le 1,\ \tfrac{1}{2} \le Y \le 1\right) &= \int_{\frac{1}{2}}^{1} \int_{\frac{1}{2}}^{\sqrt{y}} \frac{3}{2}\, dx\, dy \\ &= \int_{\frac{1}{2}}^{1} \frac{3}{2} \left(\sqrt{y} - \frac{1}{2}\right) dy \\ &= \frac{5}{8} - \left(\frac{1}{2}\right)^{3/2}; \end{aligned}$$

(d) $$\begin{aligned} P(X \ge \tfrac{1}{2}, Y \ge \tfrac{1}{2}) &= P(\tfrac{1}{2} \le X \le 1,\ \tfrac{1}{2} \le Y \le 1) \\ &= \frac{5}{8} - \left(\frac{1}{2}\right)^{3/2}. \end{aligned}$$

(e) X and Y are dependent.

(10')

4.1–12 The area of the space is

$$\int_2^6 \int_1^{14-2t_2} dt_1 dt_2 = \int_2^6 (13 - 2t_2)\, dt_2 = 20;$$

Thus

$$\begin{aligned} P(T_1 + T_2 > 10) &= \int_2^4 \int_{10-t_2}^{14-2t_2} \frac{1}{20} dt_1 dt_2 \\ &= \int_2^4 \frac{4 - t_2}{20} dt_2 \\ &= \left[-\frac{(4-t_2)^2}{40}\right]_2^4 = \frac{1}{10}. \end{aligned}$$

(10')

4.2–6 Note that X is $b(3,\ 1/6)$, Y is $b(3,\ 1/2)$ so

(a) $E(X) = 3(1/6) = 1/2,$

(b) $E(Y) = 3(1/2) = 3/2,$

(c) $\mathrm{Var}(X) = 3(1/6)(5/6) = 5/12,$

(d) $\mathrm{Var}(Y) = 3(1/2)(1/2) = 3/4;$

(e)
$$\begin{aligned}\mathrm{Cov}(X,Y) &= 0 + (1)f(1,1) + 2f(1,2) + 2f(2,1) - (1/2)(3/2)\\ &= (1)(1/6) + 2(1/8) + 2(1/24) - 3/4\\ &= -1/4;\end{aligned}$$

(f) $\rho = \dfrac{-1/4}{\sqrt{\dfrac{5}{12}\cdot\dfrac{3}{4}}} = \dfrac{-1}{\sqrt{5}}.$

(10')

4.2–12 **(a)**
$$\begin{aligned}f_1(x) &= \int_x^1 8xy\,dy = 4x(1-x^2), \qquad 0 \le x \le 1,\\ f_2(y) &= \int_0^y 8xy\,dx = 4y^3, \qquad 0 \le y \le 1;\end{aligned}$$

(b)
$$\begin{aligned}\mu_X &= \int_0^1 x4x(1-x^2)\,dx = \frac{8}{15},\\ \mu_Y &= \int (y * 4y^3\,dy = \frac{4}{5},\\ \sigma_X^2 &= \int_0^1 (x-8/15)^2 4x(1-x^2)\,dx = \frac{11}{225},\\ \sigma_Y^2 &= \int ((y-4/5)^2 * 4y^3\,dy = \frac{2}{75},\end{aligned}$$

$$\begin{aligned}\mathrm{Cov}(X,Y) &= \int_0^1\int_x^1 (x-8/15)(y-4/5)8xy\,dy\,dx = \frac{4}{225},\\ \rho &= \frac{4/225}{\sqrt{(11/225)(2/75)}} = \frac{2\sqrt{66}}{33};\end{aligned}$$

(c) $y = \dfrac{20}{33} + \dfrac{4x}{11}.$

(10')

4.3–4 **(a)** X is $b(400,\ 0.75)$;

(b) $E(X) = 300,\ \text{Var}(X) = 75$;

(c) $b(300,\ 2/3)$;

(d) $E(Y) = 200,\ \text{Var}(Y) = 200/3$.

(10')

4.3–8 **(a)** X and Y have a trinomial distribution with $n = 30,\ p_1 = 1/6,\ p_2 = 1/6$.

(b) The conditional p.d.f. of X, given $Y = y$, is

$$b\left(n - y,\ \frac{p_1}{1 - p_2}\right) = b(30 - y,\ 1/5).$$

(c) Since $E(X) = 5$ and $\text{Var}(X) = 25/6$, $E(X^2) = \text{Var}(X) + [E(X)]^2 = 25/6 + 25 = 175/6$. Similarly, $E(Y) = 5$, $\text{Var}(Y) = 25/6$, $E(Y^2) = 175/6$. The correlation coefficient is

$$\rho = -\sqrt{\frac{(1/6)(1/6)}{(5/6)(5/6)}} = -1/5$$

so

$$E(XY) = -1/5\sqrt{(25/6)(25/6)} + (5)(5) = 145/6.$$

Thus

$$E(X^2 - 4XY + 3Y^2) = \frac{175}{6} - 4\left(\frac{145}{6}\right) + 3\left(\frac{175}{6}\right) = \frac{120}{6} = 20.$$

(10')

4.3–16 **(a)** $h(y\,|\,x) = \dfrac{1}{x}, \qquad 0 < y < x,\ \ 0 < x < 1,$

(b) $E(Y\,|\,x) = \displaystyle\int_0^x \frac{y}{x}\,dy = \frac{x}{2},$

(c) $f(x, y) = h(y\,|\,x)f_1(x) = \left(\dfrac{1}{x}\right)(1) = \dfrac{1}{x}, \qquad 0 < y < x,\ \ 0 < x < 1,$

(d) $f_2(y) = \displaystyle\int_y^1 \frac{1}{x}\,dx = -\ln\ y, \qquad 0 < y < 1.$