HW#6 Solutions (10 points each)
210 points possible

4.5–2 (a)
$$P(X_1 = 2, X_2 = 4) = \left[\frac{3!}{2!1!}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^1\right]\left[\frac{5!}{4!1!}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^1\right]$$
$$= \frac{15}{2^8} = \frac{15}{256}. = 0.0586$$

(b) $\{X_1 + X_2 = 7\}$ can occur in the two mutually exclusive ways: $\{X_1 = 3, X_2 = 4\}$ and $\{X_1 = 2, X_2 = 5\}$. The sum of the probabilities of the two latter events is

$$\left[\frac{3!}{3!0!}\left(\frac{1}{2}\right)^3\right]\left[\frac{5!}{4!1!}\left(\frac{1}{2}\right)^5\right] + \left[\frac{3!}{2!1!}\left(\frac{1}{2}\right)^3\right]\left[\frac{5!}{5!0!}\left(\frac{1}{2}\right)^5\right] = \frac{5+3}{2^8} = \frac{1}{32}. = 0.03125$$

4.5–6 Let $Y = \max(X_1, X_2)$. Then

$$\begin{aligned} G(y) &= [P(X \le y)]^2 \\ &= \left[\int_1^y \frac{4}{x^5}\,dx\right]^2 \\ &= \left[1 - \frac{1}{y^4}\right]^2, \qquad 1 < y < \infty \\ g(y) &= G'(y) \\ &= 2\left(1 - \frac{1}{y^4}\right)\left(\frac{4}{y^5}\right), \qquad 1 < y < \infty; \\ &= 8/y^5 - 8/y^9 \end{aligned}$$

$$\begin{aligned} E(Y) &= \int_1^\infty y \cdot 2\left(1 - \frac{1}{y^4}\right)\left(\frac{4}{y^5}\right) dy \\ &= \int_1^\infty 8\,[y^{-4} - y^{-8}]\,dy \\ &= \frac{32}{21}. = 1.523 \end{aligned}$$

4.5–10 $P(1 < \min X_i) = [P(1 < X_i)]^3 = \left(\int_1^\infty e^{-x}\,dx\right)^3 = e^{-3} = 0.05.$

4.5–14
$$\begin{aligned}
f(x) &= 2x, \qquad 0 < x < 1; \\
F(x) &= x^2, \qquad 0 < x < 1; \\
G(y) &= P(X_1 \le y)P(X_2 \le y)P(X_3 \le y)P(X_4 \le y) \\
&= [P(X \le y)]^4 = y^8, \qquad 0 < y < 1; \\
g(y) &= G'(y) = 8y^7, \qquad 0 < y < 1; \\
E(Y) &= \textstyle\int_0^1 y\, 8y^7\, dy \\
&= \left[\frac{8}{9}y^9\right]_0^1 = \frac{8}{9}.
\end{aligned}$$

So the value in dollars is $\$(8/9)(100{,}000)$.

4.5–16
$$\begin{aligned}
P(\max > 8) &= 1 - P(\max \le 8) \\
&= \left[\sum_{x=0}^{8} \binom{10}{x} (0.7)^x (0.3)^{10-x}\right]^3 \\
&= 1 - (1 - 0.1493)^3 = 0.3844.
\end{aligned}$$

4.7–2 $\text{Var}(X) = 298 - 17^2 = 9$.

(a)
$$\begin{aligned}
P(10 < X < 24) &= P(10 - 17 < X - 17 < 24 - 17) \\
&= P(\,|X - 17| < 7) \ge 1 - \frac{9}{49} = \frac{40}{49},
\end{aligned}$$

because $k = 7/3$;

(b) $P(\,|X - 17| \ge 16) \le \dfrac{9}{16^2} = 0.035$, because $k = 16/3$.

4.7–6
$$\begin{aligned}
P(75 < \overline{X} < 85) &= P(75 - 80 < \overline{X} - 80 < 85 - 80) \\
&= P(\,|\overline{X} - 80| < 5) \ge 1 - \frac{60/15}{25} = 0.84,
\end{aligned}$$

because $k = 5/\sqrt{60/15} = 5/2$.

4.6–4
$$\begin{aligned} E(Y) &= E(X_1X_2) = E(X_1)E(X_2) = \mu_1\mu_2; \\ \mathrm{Var}(Y) &= E(X_1^2X_1^2) - (\mu_1\mu_2)^2 = E(X_1^2)E(X_2^2) - \mu_1^2\mu_2^2 \\ &= (\mu_1^2+\sigma_1^2)(\mu_2^2+\sigma_2^2) - \mu_1^2\mu_2^2 = \sigma_1^2\sigma_2^2 + \mu_1^2\sigma_2^2 + \mu_2^2\sigma_1^2. \end{aligned}$$

4.6–10
$$\begin{aligned} E[e^{tW}] &= E[e^{t(X_1+X_2+\cdots+X_h)}] = E[e^{tX_1}]E[e^{tX_2}]\cdots E[e^{tX_h}] \\ &= [1/(1-\theta t)]^h = 1/(1-\theta t)^h, \qquad t < 1/\theta, \end{aligned}$$

the moment generating function for the gamma distribution with mean $h\theta$.

4.6–12 **(a)** $M_W(t) = M_X(t)\cdot M_Y(t) = \frac{1}{12}(e^{2t}+2e^{3t}+3e^{4t}+3e^{5t}+2e^{6t}+e^{7t})$

(b) The p.m.f. of W is

$$P(W=w) = \begin{cases} \dfrac{1}{12}, & w = 2, 7, \\[2mm] \dfrac{2}{12}, & w = 3, 6, \\[2mm] \dfrac{3}{12}, & w = 4, 5. \end{cases}$$

4.6–20 Let X_1, X_2, X_3 be the number of accidents in weeks 1, 2, and 3, respectively. Then $Y = X_1 + X_2 + X_3$ is Poisson with mean $\lambda = 6$ and

$$P(Y = 7) = 0.744 - 0.606 = 0.138.$$

4.6–26 $Y = X_1 + X_2 + X_3 + X_4$ has a gamma distribution with $\alpha = 6$ and $\theta = 10$. So

$$P(Y > 90) = \int_{90}^{\infty} \frac{1}{\Gamma(6)10^6} y^{6-1} e^{-y/10}\, dy = 1 - 0.8843 = 0.1157.$$

The final answer was calculated using Minitab.

5.2–4 **(a)** 1.282; **(b)** -1.645;

(c) -1.66; **(d)** -1.82.

5.2–6 $M(t) = e^{166t+400t^2/2}$ so

(a) $\mu = 166$; **(b)** $\sigma^2 = 400$;

(c) $P(170 < X < 200) = P(0.2 < Z < 1.7) = 0.3761$; $= 0.9554 - 0.5793 = 0.3761$

(d) $P(148 \le X \le 172) = P(-0.9 \le Z \le 0.3) = 0.4338$. $= 0.6179 - 0.1841 = 0.4338$

5.2–14 **(a)** $P(X > 22.07) = P(Z > 1.75) = 0.0401$;

(b) $P(X < 20.857) = P(Z < -1.2825) = 0.10$. Thus the distribution of Y is $b(15,\ 0.10)$ and from Table II in the Appendix, $P(Y \le 2) = 0.8159$.

5.2–20 **(a)** $\Phi(0.7) - \Phi(-1.95) = 0.7580 - 0.0256 = 0.7324$;

(b) $1 - \Phi(1.55) = 0.0606$;

(c) $\Phi(-0.55) = 0.2912$.

5.2–26
$$
\begin{aligned}
P(X > 120 \mid X > 105) &= \frac{P(X > 120)}{P(X > 105)} \\
&= \frac{1 - \Phi(2)}{1 - \Phi(1)} \\
&= \frac{0.0228}{0.1587} = 0.1437.
\end{aligned}
$$

5.3–4 **(a)** $P(X < 6.0171) = P(Z < -1.645) = 0.05$;

(b) Let W equal the number of boxes that weigh less than 6.0171 pounds. Then W is $b(9,\, 0.05)$ and $P(W \le 2) = 0.9916$;

(c)
$$\begin{aligned} P(\overline{X} \le 6.035) &= P\left(Z \le \frac{6.035 - 6.05}{0.02/3}\right) \\ &= P(Z \le -2.25) = 0.0122. \end{aligned}$$

5.3–10 The distribution of Y is $N(3.54,\, 0.0147)$. Thus

$$P(Y > W) = P(Y - W > 0) = P\left(Z > \frac{-0.32}{\sqrt{0.0147 + 0.082}}\right) = 0.9830.$$

$P(Z > 2.115) = 0.9830$

5.3–18 Let $Y = X_1 + X_2 + \cdots + X_n$. Then Y is $N(800n,\, 100^2 n)$. Thus

$$\begin{aligned} P(Y \ge 10000) &= 0.90 \\ P\left(\frac{Y - 800n}{100\sqrt{n}} \ge \frac{10000 - 800n}{100\sqrt{n}}\right) &= 0.90 \\ -1.282 &= \frac{10000 - 800n}{100\sqrt{n}} \\ 800n - 128.2\sqrt{n} - 10000 &= 0. \end{aligned}$$

Either use the quadratic formula to solve for $\sqrt{n}$ or use Maple to solve for n. We find that $\sqrt{n} = 3.617$ or $n = 13.08$ so use $n = 14$ bulbs.

5.3-22

Can use transformation method

$X \sim N(0,1) \Rightarrow f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$.

$Y = |X|$

Let $x_1 = -y$ $x_1 \in (-\infty, 0)$ and $x_2 = y$ $x_2 \in (0, \infty)$

$\frac{dx_1}{dy} = -1$ $\frac{dx_2}{dy} = 1$

$f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-(-y)^2/2} \cdot |-1| + \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \cdot |1|$

$= 2/\sqrt{2\pi}\, e^{-y^2/2}$ $y \in (0, \infty)$