1 Solution to Exercise 2.1.

Problem Statement: *Give timeseries plot and lagged scatterplots at lags 1 and 2 of the given series. Guess at possible autocorrelation by inspecting the graphs.*

Solution: I downloaded the file from the website into the working directory. Here is the Splus code to read in the data and do the plots:

```
> reactor=scan("reactor.txt")
> tsplot(reactor)
> #Looks autocorrelated to me.
> length(reactor)
[1] 36
> plot(reactor[1:35],reactor[2:36],main="Lag 1 Scatterplot")
> #Clearly a positive correlation
> plot(reactor[1:34],reactor[3:36],main="Lag 2 Scatterplot")
> #maybe a positive correlation; not so obvious
```

Follow the following links to see postscript files of the plots

> output from `tsplot(reactor)`;

> output from `plot(reactor[1:35],reactor[2:36],main="Lag 1 Scatterplot")`;

> output from `plot(reactor[1:34],reactor[3:36],main="Lag 2 Scatterplot")`;

Here is a Matlab session. For some reason, there were a bunch of extra '13"s in my input which had to be eliminated. They must have been invisible end of line characters left in the file after downloading to my PC.

```
z=dlmread('c:\windows\profiles\dennis~1.cox\desktop\hw1\sdir\reactor.txt');
size(z)
```

```
ans =

    74     1

 z(1:6)

ans =

   200
    13
   202
    13
   208
    13

 z(70:74)

ans =

    13
   204
    13
    13
    13

 z=z(1:2:71);
```

```
 size(z)

ans =

    36     1

 plot(z)
 print -dbitmap
 plot(z(1:35),z(2:36),'.')
 print -dbitmap
 plot(z(1:34),z(3:36),'.')
 print -dbitmap
```

Follow the following links to see JPEG files of the plots

≪ output from `plot(z)`;

≪ output from `plot(z(1:35),z(2:36),'.')`;

≪ output from `plot(z(1:34),z(3:36),'.')`.

2 Solution to Exercise 2.2.

Problem Statement: *Are the following valid autocorrelation functions?*

(a) $\rho_1 = .8,\ \rho_2 = .55,\ \rho_k = 0$ *for* $k > 2$.

(a) $\rho_1 = .8,\ \rho_2 = .28,\ \rho_k = 0$ *for* $k > 2$.

Solution: In order for a given sequence to be a valid autocorrelation function, we must of course have $\rho_0 = 1$. We assume this is intended, as the value of ρ_0 is not explicitly stated. Furthermore, as in section 2.1.3, all of the correlation matrices

$$\begin{bmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_{n-1} \\ \rho_1 & 1 & \rho_1 & \cdots & \rho_{n-2} \\ \rho_2 & \rho_1 & 1 & \cdots & \rho_{n-3} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \rho_{n-1} & \rho_{n-2} & \rho_{n-3} & \cdots & 1 \end{bmatrix} = \mathrm{P}_n$$

must be nonnegative definite (this is a correction to what's in the book - they say positive definite) *for all* $n = 2, 3, \ldots$. This business with positive definiteness vs. nonnegative definiteness is discussed in the notes below.

On p. 28 we see the conditions

$$-1 < \rho_1 < 1, \tag{1}$$

$$-1 < \rho_2 < 1, \tag{2}$$

$$-1 < \tfrac{\rho_2 - \rho_1^2}{1 - \rho_1^2} < 1. \tag{3}$$

Actually, all the $<$ should be $\leq$ (because it is only necessary that the determinants that were evaluated to produce these be nonnegative, not positive). In fact, *these are necessary but not sufficient* for the given function to be a valid autocorrelation function since they only verify positivity of the determinants of P_n for $n = 1$, 2, and 3, not all n. We can apply these conditions to the given potential autocorrelation functions, and if any of the conditions fails, then we know the given functions are not valid autocorrelation functions. However, even if all three conditions are verified, then we cannot be sure that the given function is an autocorrelation function because we don't know about the sign of the determinant of P_n for $n > 3$.

In both cases, clearly (1) and (2) hold, so there only remains to check (3). For the putative autocorrelation in (a),

$$\frac{.55 - .8^2}{1 - .8^2} = -.25,$$

so that one is OK by (3), and for (b), the Matlab computations are

```
format long
(.28-.8^2)/(1-.8^2)


ans =


  -1.00000000000000
```

It looks like it fails the condition as stated in the book, but doesn't fail it according to the amended statement as above (and mentioned in class).

So at this point, we don't really know more than when we started. It is quite impossible to check the conditions that the determinant of P_n is nonnegative for all n. However, as mentioned in class, γ_k is a valid autocovariance if and only if

$$g(f) = \gamma_0 + 2\sum_{k=1}^{\infty} \gamma_k cos(2\pi f k) \tag{4}$$

is a nonnegative function. This is sort of alluded to in the book. About midway down p. 28, "As will be shown in Section 2.2.3, all of these conditions can be brought together in the definition of the spectrum." However, all I can find in Section 2.2.3 is the following statement in the first paragraph on p. 40, "Conversely, since the $\{\gamma_k\}$ from a positive definite sequence, it follows from Herglotz's theorem ... that a unique function p(f) exists such that ..." It doesn't actually state that nonnegativeness of the Fourier series transform as in (4) is equivalent to the nonnegative definiteness property.

Anyway, applying this fancier condition to the first alleged autocorrelation function, the corresponding spectrum (or spectral density function) is

$$g(f) \;=\; 1 \;+\; 1.6\cos(2\pi f) \;+\; 1.1\cos(4\pi f).$$

The plot of this function can be found at this link. This function clearly goes negative, so we do not have a valid autocorrelation function. For the spectral density in part(b):

$$g(f) \;=\; 1 \;+\; 1.6\cos(2\pi f) \;+\; .56\cos(4\pi f),$$

the corresponding plot is here. We see this one also goes negative in places, so we do not have a valid autocorrelation function for this one either.

Comments: I don't expect you to understand necessarily what is going on with this problem. The take home message is that it is nontrivial to determine if a given function is an autocorrelation function unless it is computed from a process known to be stationary.

Also, regarding the requirement of (strict) positive definiteness of all P_n vs. nonnegative definiteness, the latter is all that is needed, but if one of the determinants is 0, that says there is some linear combination of finitely many values of the process which is almost surely a constant (0 if the process mean is 0). This would mean that the process if perfectly predictable given a large enough block of past observations. This never occurs in practice, so the condition as stated in the text is "safe" in practice but not mathematically correct.

3 Solution to Exercise 2.3.

Problem Statement: *Two stochastic processes Z_{1t} and Z_{2t} have autocovariance functions*

$$
\begin{aligned}
Z_{1t}: &\quad \gamma_0 = 0.5, \quad \gamma_1 = .2, \quad \gamma_j = 0, \quad j \geq 2, \\
Z_{2t}: &\quad \gamma_0 = 2.30, \quad \gamma_1 = -1.43, \quad \gamma_2 = .3, \quad \gamma_j = 0, \quad j \geq 3.
\end{aligned}
$$

Calculate the autocovariance function for the process $Z_{3t} = Z_{1t} + 2Z_{2t}$ and verify that it is a valid stationary process.

Solution: Of course, we have to assume something about the *joint* distribution of the processes Z_{1t} and Z_{2t}, so that we can compute $\mathrm{Cov}[Z_{1t}, Z_{2s}]$ for all s and t. As stated in lecture, just assume that the two processes are independent in the strictly stationary case and uncorrelated in the second order case, i.e. that $\mathrm{Cov}[Z_{1t}, Z_{2s}] = 0$ for all s and t. This is typical – if people don't know ("people" meaning the statistically unsophisticated), given a couple of marginal distributions and needing a joint distribution, they assume independence. But this is just an exercise from a textbook and I assume the authors were just careless in not stating the assumption that would make the problem tractable at this stage. If we don't make some such assumption, then the problem is not do-able in the sense that we can't compute the autocovariance of the linear combination from the autocovariances of the components. Furthermore, it is entirely possible to have two processses which themselves are (second order) stationary but the vector valued process is not (second order) stationary.

But life is easy with the uncorrelatedness assumption.

Definition *Two time series type stochastic processes $\langle Z_{1t} : t = ... -2, -1, 0, 1, 2, ...\rangle$ and $\langle Z_{2t} : t = ... -2, -1, 0, 1, 2, ...\rangle$ are called uncorrelated if $Cov[Z_{1t}, Z_{2s}] = 0$ for all s and t.*

Proposition *If Z_{1t} and Z_{2t} are uncorrelated time series type stochastic processes* *which are (second order) stationary with autocovariances γ_1 and γ_2, respectively, then for any constants a_1 and a_2, $Z_t = a_1 Z_{1t} + a_2 Z_{2t}$ is a (second order) stationary process and the autocovariance function is given by*

$$\gamma(h) \;=\; Cov(Z_t, Z_{t+h}) \;=\; a_1^2 \gamma_1(h) \;+\; a_2^2 \gamma_2(h). \tag{5}$$

Proof. Of course, we have two theorems to prove, one for the strictly stationary case and one for the second order stationary case (that's why "second order" is in parentheses, in case you didn't know). What we'll do here is quickly dispense with the second order stuff (which also takes care of the formula for the autocovariance function in the strictly stationary case) and handwave the stationary case.

OK, for the second order stuff we only have to verify that the mean is constant and the autocovariance only depends on the lag. The mean, as usual, is pretty easy:

$$\begin{aligned} E[Z_t] &= E[a_1 Z_{1t} + a_2 Z_{2t}] \\ &= a_1 E[Z_{1t}] \;+\; a_2 E[Z_{2t}] \\ &= a_1 \mu_1 \;+\; a_2 \mu_2, \end{aligned}$$

where I just made up the notations $\mu_i = E[Z_{it}]$, which don't depend on t by assumption. Clearly $E[Z_t]$ is a constant, independent of t. Now the autocovariance, using properties of the covariance operator already discussed in class:

$$\begin{aligned} \gamma(h) &= \mathrm{Cov}(Z_t, Z_{t+h}) \\ &= \mathrm{Cov}\left(a_1 Z_{1t} + a_2 Z_{2t}, a_1 Z_{1,t+h} + a_2 Z_{2,t+h}\right) \\ &= \mathrm{Cov}\left(a_1 Z_{1t}, a_1 Z_{1,t+h}\right) \;+\; \mathrm{Cov}\left(a_1 Z_{1t}, a_2 Z_{2,t+h}\right) \;+ \\ &\qquad \mathrm{Cov}\left(a_2 Z_{2t}, a_1 Z_{1,t+h}\right) \;+\; \mathrm{Cov}\left(a_2 Z_{2t}, a_2 Z_{2,t+h}\right) \\ &= a_1^2 \gamma_1(h) + 0 + 0 + a_2^2 \gamma_2(h) \qquad (6) \\ &= a_1^2 \gamma_1(h) \;+\; a_2^2 \gamma_2(h). \end{aligned}$$

Here, the zeroes in (6) are from our uncorrelatedness of the process assumption. Since the autocovariance function depends only on the lag h, this completes the proof of second order stationarity and verifies the formula for the autocovariance function.

Turning to the issue of strict stationarity, assume Z_{1t} and Z_{2t} are strictly stationary and independent processes. Consider the typical case where any finite number of random variables from the process will have a joint density (i.e., has a continuous distribution, as opposed to a discrete distribution where we would have a joint probability mass function). Subscripts to the joint density will indicate what random variables it is the density of. In order to avoid too much notation, let

$$Z_i[t,h] \;=\; (Z_{it}, Z_{i,t+1}, \ldots, Z_{i,t+h}), \quad h \geq 0, \quad i = 1, 2.$$

Also, let z_i denote a vector of dimension $h+1$ which will be used as a variable. The joint density of $(Z_1[t,h], Z_2[t,h])$ is

$$\begin{aligned} & p_{Z_1[t,h],Z_2[t,h]}(z_1, z_2) \\ &= p_{Z_1[t,h]}(z_1) p_{Z_2[t,h]}(z_2) \quad (7) \\ &= p_{Z_1[0,h]}(z_1) p_{Z_2[0,h]}(z_2) \quad (8) \\ &= p_{Z_1[0,h],Z_2[0,h]}(z_1, z_2). \quad (9) \end{aligned}$$

i.e. the product of the marginals. Here, (7) and (9) follow from the independence assumption, and (8) from the stationarity assumptions on the individual processes. Thus, we see that the distribution of the vector valued process (Z_{1t}, Z_{2t}) is shift invariant, hence if g is any function of two variables, the distribution of $g(Z_{1t}, Z_{2t})$ will be shift invariant, i.e. the process will be stationary.

This completes the proof. Application to the original problem is trivial. Plugging in the values:

$$\gamma_{Z_3} \;=\; \gamma_{Z_1} + 4\gamma_{Z_2}$$

$$\gamma_{Z_3}(k) = \begin{cases} 9.7 & k = 0, \\ -5.52 & k = 1, \\ 1.2 & k = 2, \\ 0 & k > 2. \end{cases}$$

4 Solution to Exercise 2.4.

Problem Statement: *Calculate the sample autocovariance and autocorrelation function for the reactor data up to lag 2. Graph.*

Solution: Of course, we can do it for a lot more lags than 2. I made a matlab function to do an Splus style plot of the ACF. You can download the function here. A jpeg of the plot may be viewed here. The computed values at lags 1 and 2 are .4910 and .1639, respectively.

5 Solution to Exercise 2.5.

Problem Statement: *Under the supposition that $\rho_j = 0$ for $j > 2$,*

(a) *Obtain approximate standard errors for r_1, r_2, and r_j, $j > 2$.*

(b) *Obtain the approximate correlation between r_4 and r_5.*

Solution: For part (a), we apply (2.1.11), p. 32:

$$\operatorname{Var}[r_k] \approx \frac{1}{N} \sum_{v=-\infty}^{\infty} \left(\rho_v^2 + \rho_{v+k}\rho_{v-k} - 4\rho_k\rho_v\rho_{v-k} + 2\rho_v^2\rho_k^2 \right). \tag{10}$$

$$= \frac{1}{N} \left[(1+2\rho_k^2) \sum_{v=-\infty}^{\infty} \rho_v^2 + \sum_{v=-\infty}^{\infty} \rho_{v+k}\rho_{v-k} - 4\rho_k \sum_{v=-\infty}^{\infty} \rho_v\rho_{v-k} \right] \tag{11}$$

Here, (10) is right out of the book, and (11) is just a different version. Of course, the standard error is just the square root of this. There will only be finitely many terms in the sum since $\rho_k = 0$ for $k > 2$.

For $k = 1$,

$$\begin{aligned}
&\operatorname{Var}[r_1] \\
&\approx \frac{1}{N} \left[(1+2\rho_1^2) \sum_{v=-\infty}^{\infty} \rho_v^2 + \sum_{v=-\infty}^{\infty} \rho_{v+1}\rho_{v-1} - 4\rho_1 \sum_{v=-\infty}^{\infty} \rho_v\rho_{v-1} \right] \\
&= \frac{1}{N} \left[(1+2\rho_1^2) \sum_{v=-2}^{2} \rho_v^2 + \sum_{v=-1}^{1} \rho_{v+1}\rho_{v-1} - 4\rho_1 \sum_{v=-1}^{2} \rho_v\rho_{v-1} \right] && (12) \\
&= \frac{1}{N} \left[(1+2\rho_1^2)\left(1+2\rho_1^2+2\rho_2^2\right) + \left(\rho_1^2 + 2\rho_2\right) - 4\rho_1 \left(2\rho_1\rho_2 + 2\rho_1\right) \right] && (13) \\
&= \frac{1}{N} \left[1 - 3\rho_1^2 + 2\rho_2^2 + 4\rho_1^4 + 4\rho_1^2\rho_2^2 + 2\rho_2 - 8\rho_1^2\rho_2 \right]. && (14)
\end{aligned}$$

Here, (12) is obtained by eliminating the summands which we know are 0 from the assumption $\rho_k = 0$, $k > 2$, (13) is just all the summations expanded out, and (14) was obtained by "simplifying" the expression in Maple:

```
> s:=(1+2*r1^2)*(1+2*r1^2+2*r2^2)+(r1^2+2*r2)-4*r1*(2*r1*r2+2*r1);
                   2               2         2        2
   s := (1 + 2 r1 ) (1 + 2 r1  + 2 r2 ) + r1  + 2 r2 - 4 r1 (2 r1 r2 + 2 r1)
```

```
> simplify(s);
                         2         2         4         2    2                2
            1 - 3 r1  + 2 r2  + 4 r1  + 4 r1  r2  + 2 r2 - 8 r1  r2
```

Now, for $k=2$,

$$\begin{aligned}
\operatorname{Var}[r_2] \quad &\approx \quad \frac{1}{N}\left[(1+2\rho_2^2)\sum_{v=-\infty}^{\infty}\rho_v^2 \;+\; \sum_{v=-\infty}^{\infty}\rho_{v+2}\rho_{v-2} \;-\; 4\rho_2\sum_{v=-\infty}^{\infty}\rho_v\rho_{v-2}\right] \\
&= \quad \frac{1}{N}\left[(1+2\rho_2^2)\sum_{v=-2}^{2}\rho_v^2 \;+\; \sum_{v=0}^{0}\rho_{v+2}\rho_{v-2} \;-\; 4\rho_2\sum_{v=0}^{2}\rho_v\rho_{v-2}\right] && (15) \\
&= \quad \frac{1}{N}\left[(1+2\rho_2^2)\left(1+2\rho_1^2+2\rho_2^2\right) \;+\; \rho_2^2 \;-\; 4\rho_2\left(2\rho_2+\rho_1^2\right)\right] && (16) \\
&= \quad \frac{1}{N}\left[1+4\rho_1^2-5\rho_2^2+4\rho_1^4+4\rho_1^2\rho_2^2-4\rho_1^2\rho_2\right] && (17)
\end{aligned}$$

The steps are all the same as before. Here is the Maple code for the simplification:

```
> t:=(1+2*r1^2)*(1+2*r1^2+2*r2^2)+r2^2-4*r2*(2*r2+r1^2);
                         2              2         2       2             2
        t := (1 + 2 r1 ) (1 + 2 r1  + 2 r2 ) + r2  - 4 r2 (r1  + 2 r2)
```

```
> simplify(t);
                            2         2         4         2    2         2
               1 + 4 r1  - 5 r2  + 4 r1  + 4 r1  r2  - 4 r1  r2
```

Finally, the last part of (a) is $k>2$. By supposition, $\rho_k=0$, so we can immediately drop the terms $2\rho_k^2\sum_{v=-\infty}^{\infty}\rho_v^2$ and $4\rho_k\sum_{v=-\infty}^{\infty}\rho_v\rho_{v-k}$ from (11). Also, for any integer v, at least one of $v+k$ or $v-k$ is outside the interval $[-2,2]$, so the term $\sum_{v=-\infty}^{\infty} \rho_{v+k}\rho_{v-k}$ is also 0. Thus,

$$
\begin{aligned}
\mathrm{Var}[r_k] &\approx \frac{1}{N}\sum_{v=-\infty}^{\infty} \rho_v^2 \\
&= \frac{1}{N}\left(1 + 2\rho_1^2 + 2\rho_2^2\right) \qquad (18)
\end{aligned}
$$

Finally, for part (b), we use (2.1.14), p. 34. Since 4 and 5 are bigger than 2, this formula applies (note that it only applies once we are into the region where $k > q$ and q is the lag of the last nonzero autocorrelation). We have

$$
\begin{aligned}
\mathrm{Cov}[r_4, r_5] &\approx \frac{1}{N}\sum_{v=-2}^{2} \rho_v \rho_{v+1} \\
&= \frac{1}{N}\left(2\rho_2\rho_1 + 2\rho_1\right) \qquad (19) \\
&= \frac{1}{N} 2\rho_1\left(\rho_2 + 1\right) \qquad (20)
\end{aligned}
$$

Remarks: The takehome message is that these formulae exist. If you are like me and aren't satisfied until you know where they come from, you can take our advanced time series class. The chances of these formulae ever coming up on an exam are two: slim and none.

Remember: these results are valid for *Gaussian* processes. Under a "long range" independence assumption, one can derive formulae for the approximate variances (variances of asymptotic normal distributions) which depend on the first four moments, basically. The particular algebraic forms are highly dependent on the moments of multivariate normal distributions.

6 Solution to Exercise 2.6.

Problem Statement: *"Using the data of Exercise 2.1, calculate the periodogram for the periods 36, 18, 12, 9, 36/5, 6 and draw up an analysis of variance table showing the mean squares associated with these periods and the residual mean square."*

Solution: Don't you just love the Brits and the way that they cling to these archaic usages. Not. I guess I had enough people in my office about this one: just get the periodogram, test if the frequencies associated with the periods mentioned are significantly larger than for the other frequencies (these are the frequencies used for computing the "residual mean square").

There are 36 degrees of freedom in the data, but we lose 1 from the mean (DC component, if you are an ECE major), leaving 35. The problem axes us to test for significant size of frequencies 1/36, 1/18 = 2/36, 1/12 = 3/36, 1/9 = 4/36, 5/36, and 1/6 = 6/36, which just happen to be the first 6 (nonzero) Fourier frequencies ("fooyer freaks", in street talk, where the subtleties of the French language are not appreciated). Basically, they want us to check for a low-frequency signal in these data. The 6 values of the periodogram have 12 degrees of freedom (2 each) and the and the other 12 "residual" values in the periodogram (corresponding to the higher frequencies) have 23 degrees of freedom. This can be seen in two ways: $23 = 35 - 12$, the total degrees of freedom minus the degrees of freedom in the low frequencies being tested, and in the 12 "residual"values, 11 have 2 degrees of freedom each and the last one has only 1 degree of freedom (this is always the case with an even sample size that the highest frequency only has a real or cosine part and hence only 1 degree of freedom), so $2 \times 11 + 1 = 23$.

Thus, our test statistic will be

$$F \;=\; \frac{(\text{sum of first 6 periodogram values})/12}{(\text{sum of last 12 periodogram values})/6}.$$

If the null hypothesis (that the values come from a Gaussian white noise) is true, then the sampling distribution of this test statistic has an F-distribution with $(12, 23)$ degrees of freedom. The null hypothesis is rejected if F is too large. The 5% critical value in the upper tail of the F distribution with $(12, 23)$ degrees of freedom is 2.2036, so we reject the null hypothesis at the .05 level of significance if the observed value of the test statistic is larger than 2.2036. We can compute a p-value by computing the the area under the density of the $F(12, 23)$ distribution to the right of the observed value, and reject at the .05 level of significance if the p-value is less than .05 (reject at any level α if the p-value is less than α).

Here is the computation in Splus. I had already "scanned" in the data into `reactor`.

```
> spec=spec.pgram(reactor,taper=0,pad=0,detrend=F,demean=T)
> length(spec$spec)
[1] 19
> spec$freq[2]
[1] 0.02777778
> 1/36
[1] 0.02777778
> #No extra padding -- That's Good!!!
> pgram=10^(.1*spec$spec)
> plot(pgram)
> #Boy, that 3'rd frequency has a lot of energy
> test=(sum(pgram[2:7])/12)/(sum(pgram[8:19])/23)
> test
[1] 3.635956
> 1-pf(test,12,23)
[1] 0.003814983
```

```
> #OK, it's statistically significant.  There's something going
> #on in the 1st 6 fourier freq's.
```

The p-value is about 0.0038, which is pretty highly statistically significant. Yes, Virginia, there is a low frequency signal in these data.

Here's the corresponding calculation in Matlab, using the `pdgrm` function I created and using `fcdf` (the cumulative distribution function of the F-distribution) from the statistics toolbox (not available everywhere).

```
 pgram=pdgrm(z);
 test=(sum(pgram(1:6))/12)/(sum(pgram(7:18))/23);
 test

test =

    3.6360
 1-fcdf(test,12,23)

ans =

    0.0038
```

Same p-value. The plot of the periodogram is shown here.

Axe me if you don't know what a p-value is.

7 Solution to Exercise 2.7 *The Last One!!!.*

Problem Statement: *A circular stochastic process with period N is defined by $Z_t = Z_{t+N}$, for all t.*

(a) *Show that when $N = 2n$, the latent roots of the $N \times N$ autocorrelation matrix of Z_t are*

$$\lambda_k \;=\; 1 \;+\; 2\sum_{i=1}^{n-1} \rho_i \cos\left(\frac{\pi i k}{n}\right) \;+\; \rho_n \cos(\pi k), \tag{21}$$

for $k = 1, 2, \ldots, N$, and that the latent vectors corresponding to λ_k, λ_{N-k} (with $\lambda_k = \lambda_{N-k}$) are

$$\boldsymbol{\ell}_k \;=\; \left(\cos\left[\frac{\pi k}{n}\right], \cos\left[\frac{2\pi k}{n}\right], \ldots, \cos\left[2\pi k\right]\right) \tag{22}$$

$$\boldsymbol{\ell}_{N-k} \;=\; \left(\sin\left[\frac{\pi k}{n}\right], \sin\left[\frac{2\pi k}{n}\right], \ldots, \sin\left[2\pi k\right]\right). \tag{23}$$

(b) *Verify that as $N \to \infty$, with k/N fixed, λ_k tends to $g(k/N)/2$, where $g(f)$ is the spectral density function, showing that in the limit the latent roots of the autocorrelation matrix trace out the spectral curve.*

Solution: This problem is probably the most difficult of the assignment. I contemplated not assigning it, but left it in in a foolish moment of confidence in the abilities of Rice students (not to detract from that). One problem was with the quaint British terminology of "latent root" and "latent vector." We colonists know them more by their Anglicized ("Americanized(?)") German terms of "eigenvalues" and "eigenvectors." Then several students struggled with the sine-cosine stuff and I hinted "complex exponential" and they went away and never came back, probably because they eventually gave up. But then the real corker is figuring out how to get the correlation matrix written down correctly so that it incorporates the information that the process is circular. Since $Z_t = Z_{t+N} = Z_{t-N}$, we have

$$\rho_k \;=\; \mathrm{Corr}[Z_t, Z_{t+k}] \;=\; \mathrm{Corr}[Z_{t+N}, Z_{t+k}] \;=\; \rho_{N-k}. \tag{24}$$

Therefore, there are really only $n-1$ different values. Also, we have for instance

$$\begin{aligned} \rho_{n+k} &= \rho_{N-(n+k)} \\ &= \rho_{2n-n-k} \\ &= \rho_{n-k}. \end{aligned}$$

So, when we get to ρ_n in a row of the correlation matrix, we start going backwards in the subscript.

The $N \times N$ correlation matrix for $(Z_1, \ldots, Z_N)$ is then

$$\mathrm{R} = \begin{bmatrix} 1 & \rho_1 & \rho_2 & \cdots & \rho_{n-1} & \rho_n & \rho_{n-1} & \cdots & \rho_2 & \rho_1 \\ \rho_1 & 1 & \rho_1 & \cdots & \rho_{n-2} & \rho_{n-1} & \rho_n & \cdots & \rho_3 & \rho_2 \\ \rho_2 & \rho_1 & 1 & \cdots & \rho_{n-3} & \rho_{n-2} & \rho_{n-1} & \cdots & \rho_4 & \rho_3 \\ . & . & . & \cdots & . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . & . & \cdots & . & . \\ . & . & . & \cdots & . & . & . & \cdots & . & . \\ \rho_1 & \rho_2 & \rho_3 & \cdots & \rho_n & \rho_{n-1} & \rho_{n-2} & \cdots & \rho_1 & 1 \end{bmatrix} \tag{25}$$

Note how each row is obtained from the previous one by a cyclic permutation wherein the last element is taken off and put in the first position. This is the property of being a circulant matrix (just for future reference). Now consider the complex vector

$$\boldsymbol{v}_k = (\exp[2\pi ik/N], \exp[4\pi ik/N], \exp[6\pi ik/N], \ldots, \exp[2N\pi ik/N]) \tag{26}$$

$$= \boldsymbol{\ell}_k + i\boldsymbol{\ell}_{N-k}. \tag{27}$$

To complete the proof, we will show that

$$\boldsymbol{R}\boldsymbol{v}_k = \lambda_k \boldsymbol{v}_k. \tag{28}$$

Since $\boldsymbol{R}$ and λ_k are real, there is no mixing of real and imaginary parts in either side of (28), so we may match up the real and imaginary parts using (27) to conclude

$$\boldsymbol{R}\boldsymbol{\ell}_k = \lambda_k \boldsymbol{\ell}_k, \quad \boldsymbol{R}\boldsymbol{\ell}_{N-k} = \lambda_k \boldsymbol{\ell}_{N-k},$$

which proves the claims about the "latent" vectors and their corresponding "latent" roots.

To prove (28), consider the first entry of $\boldsymbol{Rv}_k$, i.e.

$$1 \cdot \exp[2\pi ik/N] + \rho_1 \cdot \exp[4\pi ik/N] + \rho_2 \cdot \exp[6\pi ik/N] + \cdots$$
$$+\rho_{n-1} \cdot \exp[2n\pi ik/N] + \rho_n \cdot \exp[2(n+1)\pi ik/N] + \rho_{n-1} \cdot \exp[2(n+2)\pi ik/N] + \cdots$$
$$+\rho_2 \cdot \exp[2(N-1)\pi ik/N] + \rho_1 \cdot \exp[2N\pi ik/N] \tag{29}$$

$$\begin{aligned} = \; & \exp[2\pi ik/N]\,\{1 + \rho_1\,(\exp[2\pi ik/N] + \exp[2(N-1)\pi ik/N]) + \\ & \rho_2\,(\exp[4\pi ik/N] + \exp[2(N-2)\pi ik/N]) + \cdots \\ & +\rho_{n-1}\,(\exp[2(n-1)\pi ik/N] + \exp[2(N-(n-1))\pi ik/N]]) + \rho_n \exp[2n\pi ik/N]\} \end{aligned} \tag{30}$$

$$\begin{aligned} = \; & \exp[2\pi ik/N]\,\{1 + \rho_1\,(\exp[2\pi ik/N] + \exp[-2\pi ik/N]) + \\ & \rho_2\,(\exp[4\pi ik/N] + \exp[-4\pi ik/N]) + \cdots \\ & +\rho_{n-1}\,(\exp[2(n-1)\pi ik/N] + \exp[-2(n-1)\pi ik/N]]) + \rho_n \exp[2n\pi ik/N]\} \end{aligned} \tag{31}$$

$$\begin{aligned} = \; & \exp[2\pi ik/N]\,\{1 + 2\rho_1 \cos[2\pi k/N] + 2\rho_2 \cos[4\pi k/N] + \cdots \\ & +\rho_{n-1} \cos[2(n-1)\pi k/N] + \rho_n \cos[\pi k]\} \end{aligned} \tag{32}$$

$$= \lambda_k \exp[2\pi ik/N]. \tag{33}$$

Justifications:

(29) This follows from the definition of matrix multiplication: the first row of $\boldsymbol{R}$ in (25) inner product with $\boldsymbol{v}_k$.

(30) We've factored out $\exp[2\pi ik/N]$ from all terms and collected the terms with a common ρ_k and factored out that ρ_k. Also, we rewrote some of the complex exponentials, viz.

$$\exp[2(n+1)\pi ik/N] \;=\; \exp[2(N-(n-1))\pi ik/N]], \tag{34}$$

which follows since $N = 2n$.

(31) Of course, $\exp[2\pi Nijk/N] = 1$ for any integers j, k. This was substituted into (30) to obtain (31).

(32) This follows from $\exp[i\theta] + \exp[-i\theta] = 2\cos\theta$. Also, $\exp[2n\pi ik/N] = \exp[\pi ik]$ $= \cos[\pi k] = (-1)^k$.

(33) This is clear from the previous line and the definition of λ_k in (21). Note that (33) is λ_k times the first entry in $\boldsymbol{v}_k$.

Now let's move on to the second component of $\boldsymbol{R}\boldsymbol{v}_k$. This is

$$\begin{gathered}\rho_1 \cdot \exp[2\pi ik/N] + 1 \cdot \exp[4\pi ik/N] + \rho_1 \cdot \exp[6\pi ik/N] + \cdots \\ +\rho_{n-2} \cdot \exp[2n\pi ik/N] + \rho_{n-1} \cdot \exp[2(n+1)\pi ik/N] + \rho_n \cdot \exp[2(n+2)\pi ik/N] + \cdots \\ +\rho_3 \cdot \exp[2(N-1)\pi ik/N] + \rho_2 \cdot \exp[2N\pi ik/N]\end{gathered} \tag{35}$$

$$\begin{gathered}= \;\exp[4\pi ik/N]\,\{1 + \rho_1\,(\exp[-2\pi ik/N] + \exp[2\pi ik/N]) + \\ \rho_2\,(\exp[4\pi ik/N] + \exp[2(N-2)\pi ik/N]) + \cdots \\ +\rho_{n-1}\,(\exp[2(n-1)\pi ik/N] + \exp[2(N-(n-1))\pi ik/N]]) + \rho_n \exp[2n\pi ik/N]\}\end{gathered} \tag{36}$$

$$= \exp[4\pi ik/N]\,\{1 + 2\rho_1 \cos[2\pi k/N] + 2\rho_2 \cos[4\pi k/N] + \cdots$$

$$+\rho_{n-1} \cos[2(n-1)\pi k/N] + \rho_n \cos[\pi k]\} \tag{37}$$

$$= \lambda_k \exp[4\pi ik/N]. \tag{38}$$

The justifications are pretty much the same. Note that at (36) we factored out $\exp[4\pi ik/N]$ from everything since this is the second component of $\boldsymbol{v}_k$ and we want to see something (namely λ_k) times it in the end (i.e. (38).

You are probably getting the general idea by now. Can we make a formal mathematical argument? Let

$$\boldsymbol{r}_i = (\rho_{i-1}, \rho_{i-2}, \ldots, \rho_i)$$

denote the $N \times 1$ vector which is the transpose of the i'th row of $\boldsymbol{R}$. Of course, $\rho_0 = 1$, $\rho_{-k} = \rho_k$, and $\rho_{k \pm N} = \rho_k$. Let

$$\boldsymbol{M} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \ldots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \ldots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \ldots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

be the cyclic $N \times N$ permutation matrix which takes the last element off an N-vector and moves it to the front. Then

$$\boldsymbol{r}_i = \boldsymbol{M}^{i-1}\boldsymbol{r}_1 = \boldsymbol{M}^i\boldsymbol{r}_0,$$

where $\boldsymbol{r}_0$ is defined in the obvious way. Note that the transpose $\boldsymbol{M}^T$ is the inverse, i.e. it moves the first element to the end of the vector. Also,

$$\begin{aligned}
\boldsymbol{M}^T\boldsymbol{v}_k &= (\exp[4\pi ik/N], \exp[6\pi ik/N], \ldots, \exp[2N\pi ik/N], \exp[2\pi ik/N]) \\
&= (\exp[4\pi ik/N], \exp[6\pi ik/N], \ldots, \exp[2N\pi ik/N], \exp[2(N+1)\pi ik/N]) \\
&= \exp[2\pi ik/N](\exp[2\pi ik/N], \exp[4\pi ik/N], \ldots, \exp[2(N-1)\pi ik/N], \exp[2N\pi ik/N]) \\
&= \exp[2\pi ik/N]\boldsymbol{v}_k.
\end{aligned}$$

The j'th entry of $\boldsymbol{R}\boldsymbol{v}_k$ is

$$\begin{aligned}
\boldsymbol{r}_j^T\boldsymbol{v}_k &= \boldsymbol{r}_0^T(\boldsymbol{M}^T)^j\boldsymbol{v}_k \\
&= \boldsymbol{r}_0^T(\exp[2j\pi ik/N]\boldsymbol{v}_k) \\
&= \; = (\boldsymbol{r}_0^T\boldsymbol{v}_k)\exp[2j\pi ik/N],
\end{aligned}$$

which is just the j'th entry of $(\boldsymbol{r}_0^T\boldsymbol{v}_k)\boldsymbol{v}_k$. This shows that multiplying $\boldsymbol{R}$ times $\boldsymbol{v}_k$ gives $\boldsymbol{v}_k$ back again but multiplied by $(\boldsymbol{r}_0^T\boldsymbol{v}_k)$. We have already worked out that $(\boldsymbol{r}_0^T\boldsymbol{v}_k) = \lambda_k$, so we have completed a fairly rigorous proof of (28).

Part (b) sort of seems trivial at this point. If $N \to \infty$ but $k/N = f$ is always fixed, then of course

$$\begin{aligned}
\lambda_k &= 1 + 2\sum_{i=1}^{N/2-1} \rho_i \cos\left(\frac{2\pi ik}{N}\right) + \rho_{N/2}\cos(\pi k) \\
&= 1 + 2\sum_{i=1}^{N/2-1} \rho_i \cos(2\pi if) + \rho_{N/2}\cos(\pi k) \\
&\to 1 + 2\sum_{i=1}^{\infty} \rho_i \cos(2\pi if) \\
&= g(f)/2,
\end{aligned}$$

where $g(f)$ is given in (2.2.13), p. 40. Clearly, we are assuming that $\rho_{N/2} \to \infty$ as $N \to \infty$.

Well, this is all a little hard to swallow, because the process was circular with period N (remember that $Z_t = Z_{t+N}$ for all N which gave us that $\rho_k = \rho_{k+N}$ for all k), so we can't be talking about the same process for all N, i.e. there is a different process for each N so all the ρ_k's need an additional subscript or something to show they depend on N, and then they would be involved in taking the limit as $N \to \infty$. Hmm. What's it all about? Well, one can actually make sense of what the authors are getting at at this point. Let Z_t be a fixed stationary process satisfying a few nice properties (e.g., that $\sum_{k=1}^{\infty} |\rho_k| < \infty$). For each N construct a circular process of period N from Z_t by

$$Z_t^{(N)} = Z_{(t+\kappa) \mathrm{mod} N}$$

where $s\mathrm{mod}N$ is the remainder on division of s by N and κ is a random number which is uniformly distributed on $0, 1, \ldots, N-1$ (discrete uniform distribution). Here, we basically take Z_0, Z_1, ..., Z_{N-1} and repeat them over and over as a block, and then give a random starting point κ. This creates a circular process with period N that looks a lot like the original Z_t. In particular, if ρ_k is the ACF for the original Z_t, then the lag 1 ACF for $Z_t^{(N)}$ is $(1 - 1/N)\rho_1 + (1/N)\rho_{N-1}$ where the perturbation comes from sticking the value of Z_{N-1} next to Z_0 in forming $Z_t^{(N)}$. Similarly, the lag 2 ACF for $Z_t^{(N)}$ is $(1 - 2/N)\rho_2 + (1/N)(\rho_{N-1} + \rho_{N-2})$, etc. Anyway, these modified ACF values will converge to the originals as $N \to \infty$. 'Nuff said.