1 Solution to Exercise 3.3.

Problem Statement: *For each of the models of exercise 3.1 and also for the following models, state whether it is* (a) *stationary;* (b) *invertible.*

Solution: These are all ARMA models, so stationarity holds if and only if the roots of the AR equation $\phi(w) = 0$ are all outside the unit circle, and invertibility if and only if the roots of the MA equation $\theta(w) = 0$ are all outside the unit circle.

Note: The authors write $\bar{z}_t$ all the time to emphasize that you have to take out the mean for these models. We will just write Z_t and assume everything is mean 0.

Model 1:

$$Z_t - 0.6Z_{t-1} = a_t.$$

The root(s) of the autoregressive characteristic equation

$$1 - 0.6v = 0,$$

is(are) $v = 1/.6 = 5/3 > 1$, outside the unit circle. Therefore, the process *is stationary.*

The root(s) of the moving average characteristic equation

$$1 = 0,$$

form an empty set, thus all roots are vacuously outside the unit circle. Put differently (in the language that was used in lecture), there are no roots of $\phi(v) = 0$ on or in the unit circle. Therefore, the process *is invertible.*

Model 2:

$$Z_t = a_t - 1.3a_{t-1} + 0.4a_{t-2}.$$

The root(s) of the autoregressive characteristic equation

$$1 = 0,$$

form an empty set, thus all roots are vacuously outside the unit circle. Put differently (in the language that was used in lecture), there are no roots of $\phi(v) = 0$ on or in the unit circle. Therefore, the process *is stationary.*

The roots of the moving average characteristic equation

$$1 - 1.3v + 0.4v^2 \;=\; 0,$$

can be determined by factoring:

$$(1 - .5v)(1 - .8v) \;=\; 0$$

which gives

$$v_1 = 1/.5 = 2, \quad v_2 = 1/.8 = 1.25.$$

Both roots are outside the unit circle. Therefore, the process *is invertible.*

<u>Model 3:</u>

$$Z_t - 0.6Z_{t-1} \;=\; a_t - 1.3a_{t-1} + 0.4a_{t-2}.$$

The root of the autoregressive characteristic equation

$$1 - 0.6v \;=\; 0,$$

is $v \;=\; 1/.6 \;=\; 5/3 > 1$, outside the unit circle. Therefore, the process *is stationary.*

The moving average operator is the same as in Model 2, so the process is invertible.

<u>Model 4:</u>

$$Z_t - 1.5Z_{t-1} + 0.6Z_{t-2} \;=\; a_t.$$

The roots of the autoregressive characteristic equation

$$1 - 1.5v + 0.6v^2 \;=\; 0,$$

are

$$\begin{aligned} v_{\pm} &= \frac{1.5 \pm \sqrt{2.25 - 2.4}}{1.2} \\ &= 1.25 \pm i\sqrt{15}/12. \end{aligned}$$

The modulus squared of these complex conjugate roots is

$$|v_+|^2 \;=\; |v_-|^2 \;=\; 1.25^2 + (\pm\sqrt{15}/12)^2 \;>\; 1.25^2 \;>\; 1,$$

outside the unit circle. Therefore, the process *is stationary.* (One can determine this without computing the roots, once it is known that the roots are complex conjugates. Recall that the product of reciprocal roots is the modulus squared and equal to the coefficient of v^2, namely 0.6 in this case, so the modulus squared is $1/0.6 > 1$.)

The process *is invertible* as in Model 1.

Model 5:

$$Z_t - Z_{t-1} \;=\; a_t - 0.5a_{t-1}.$$

The root of the autoregressive characteristic equation

$$1 - v \;=\; 0,$$

is $v \;=\; 1$, on the unit circle. Therefore, the process *is not stationary.*

The root of the moving average characteristic polynomial

$$1 - 0.5v = 0,$$

is $v = 2$, outside the unit circle. Therefore, the process *is invertible.*

Model 6:

$$Z_t - Z_{t-1} \;=\; a_t - 1.3a_{t-1} + 0.3a_{t-2}.$$

The root of the autoregressive characteristic equation

$$1 - v \;=\; 0,$$

is $v \;=\; 1$, on the unit circle. Therefore, the process *is not stationary.*

The roots of the moving average characteristic equation

$$1 - 1.3v + 0.3v^2 \;=\; 0,$$

can be determined by factoring:

$$(1 - .3v)(1 - v) \; = \; 0$$

which gives

$$v_1 = 1/.3 = 3\frac{1}{3}, \quad v_2 = 1.$$

One root is outside the unit circle but one is on the unit circle. Therefore, the process *is not invertible.*

2 Solution to Exercise 3.5.

Problem Statement:

(a) *Write down the Yule-Walker equations for Models 1 and 4.*

(b) *Solve these equations to obtain ρ_1 and ρ_2 for the two models.*

(c) *Obtain the partial autocorrelation function for the two models.*

Solution:

Model 1: Since $p = 1$, the Yule-Walker equations from (3.2.6), p. 57, reduce to

$$\rho_1 = \phi_1.$$

Of course, we may use the more general recursive form (as discussed in lecture): starting with the AR(1) equation defining the model:

$$Z_t - 0.6 Z_{t-1} = a_t,$$

we multiply by Z_{t-k} for $k > 0$ and use uncorrelatedness of Z_{t-k} and a_t to get

$$\rho_k - 0.6 \rho_{k-1} = 0,$$

or

$$\rho_k = 0.6 \rho_{k-1},$$

and hence $\rho_1 = 0.6$,

$$\rho_k = (0.6)^k, \quad k > 0.$$

In particular,

$$\rho_2 = (0.6)^2 = 0.36.$$

As in (3.2.33), p. 66,

$$\phi_{11} = \rho_1 = 0.6.$$

Also, since this process is an AR(1), we have

$$\phi_{kk} = 0, \quad k > 1.$$

Model 4. From (3.2.25), p. 61, the Yule-Walker equations are

$$\begin{aligned}\rho_1 &= 1.5 - 0.6\rho_1 \\ \rho_2 &= 1.5\rho_1 - 0.6\end{aligned}$$

and the solution (as given in (3.2.27), p. 62) is

$$\begin{aligned}\rho_1 &= \frac{1.5}{1+0.6} = 0.9375, \\ \rho_2 &= -0.6 + \frac{1.5^2}{1+0.6} = 0.80625.\end{aligned}$$

The formula for the PACF of on p. 66 yield

$$\begin{aligned}\phi_{11} &= \rho_1 = 0.9375, \\ \phi_{22} &= \frac{\rho_2 - \rho_1^2}{1-\rho_1^2} \\ &= \phi_2 \text{ (see (3.2.26), p. 62)} \\ &= -0.6, \\ \phi_{kk} &= 0, \quad k > 2.\end{aligned}$$

3 Solution to Exercise 3.6.

Problem Statement: *For the AR(2) process*

$$Z_t - Z_{t-1} + 0.5Z_{t-2} = a_t,$$

(a) *Calculate ρ_1.*

(b) *Using ρ_0 and ρ_1 as starting values and the difference equation form for the ACF, calculate the values of ρ_k for $2 \leq k \leq 15$.*

(c) *Use the plotted function to estimate the period and damping factor for the ACF.*

(d) *Check the values in (c) by direct calculation using the values of ϕ_1 and ϕ_2.*

Solution: First of all, let's check for stationarity. I know that wasn't explicitly stated to do, but the rest of the problem doesn't make sense without it. So we find the roots for

$$1 - w + .5w^2 = 0,$$

which are

$$w_{\pm} = \frac{1 \pm \sqrt{1 - 4 * (.5)}}{2 * .5} = 1 \pm i,$$

which are complex conjugate roots of modulus $\sqrt{2} > 1$, outside the unit circle.

OK, now to part (a). By (3.2.27), p. 62,

$$\rho_1 = \frac{\phi_1}{1 - \phi_2} = \frac{1}{1 - (-.5)} = 2/3.$$

For part (b), the difference equation from (3.2.19), p. 60 is

$$\rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2} = \rho_{k-1} - .5\rho_{k-2}, \quad k > 0.$$

So, for instance,

$$\rho_2 = \rho_1 - .5\rho_0 = 2/3 - 1/2 = 1/6,$$

$$\begin{aligned}
\rho_3 &= \rho_2 - .5\rho_1 \;=\; 1/6 - 1/3 \;=\; -1/6, \\
\rho_4 &= \rho_3 - .5\rho_2 \;=\; -1/6 - 1/12 \;=\; -1/4, \\
\rho_5 &= \rho_4 - .5\rho_3 \;=\; -1/4 + 1/12 \;=\; -1/6, \\
\rho_6 &= \rho_5 - .5\rho_4 \;=\; -1/6 + 1/8 \;=\; -1/24, \\
\rho_7 &= \rho_6 - .5\rho_5 \;=\; -1/24 + 1/12 \;=\; 1/24, \\
\rho_8 &= \rho_7 - .5\rho_6 \;=\; 1/24 + 1/48 \;=\; 1/16, \\
\rho_9 &= \rho_8 - .5\rho_7 \;=\; 1/16 - 1/48 \;=\; 1/24, \\
\rho_{10} &= \rho_9 - .5\rho_8 \;=\; 1/24 - 1/32 \;=\; 1/96, \\
\rho_{11} &= \rho_{10} - .5\rho_9 \;=\; 1/96 - 1/48 \;=\; -1/96 \\
\rho_{12} &= \rho_{11} - .5\rho_{10} \;=\; -1/96 - 1/192 \;=\; -1/64 \\
\rho_{13} &= \rho_{12} - .5\rho_{11} \;=\; -1/64 + 1/192 \;=\; -1/96 \\
\rho_{14} &= \rho_{13} - .5\rho_{12} \;=\; -1/96 + 1/128 \;=\; -1/384 \\
\rho_{15} &= \rho_{14} - .5\rho_{13} \;=\; -1/384 + 1/192 \;=\; 1/384.
\end{aligned}$$

Of course, you may not have had so much patience computing all those fractions. It's easy enough to write a little program to do it. Here is my Splus program, which I wrote to generate the plots on the AR(2) web page (here). Anyway, I also wrote some Matlab code to do it:

```
 phi1=1;
 phi2=-.5;
 rho=[1 2/3 ];
 for k=2:15
rhok=phi1*rho(k)+phi2*rho(k-1);
rho=[rho rhok];
end
```

```
 plot(0:15,rho,'o')
 hold on
 plot([0 15] , [0 0], 'r')
 for i=0:15
plot([i i],[0 rho(i+1)])
end
 xlabel('Lag')
 ylabel('rho')
 title('Plot of ACF for Ex. 3.6')
```

The plot is shown below.

(fig1 goes here)

From this I would estimate the period at 8. The first time the ACF is negative is at lag 3, then positive at lag 7, then negative again at lag 11 (8 units after lag 3). For the damping factor, let's look at the peak values (where the damped cosine wave has a cosine factor of about 1). We see at lag 4 a negative peak of $-1/4$, then at lag 8 a positive peak of $1/16$, then at lag 12 a negative peak of -1/64. So, it looks like ρ_k is damped by a factor of 1/4 in 4 lags, so the damping factor should be about $(1/4)^{1/4}$ $= 1/\sqrt{2}$.

OK, let's see how it squares with the computation. We know the ACF is of the form

$$\rho_k \ = \ A_1 G_1^k \ + \ A_2 G_2^k,$$

where G_i^{-1} are the roots of the characteristic polynomial and the A_i are coefficients determined from initial conditions (see e.g. (3.2.20), p. 60). In this case, we already know they are complex conjugate roots, so (3.2.21), p. 60 applies and we can write the ACF in the form

$$\rho_k \ = \ D^k \sin(2\pi f_0 k + F)/\sin F,$$

where D is the damping factor, f_0 the frequency, and F the phase. Here,

$$D \; = \; |G_i| \; = \; 1/|1 \pm i| \; = \; 1/\sqrt{2},$$

and

$$2\pi f_0 \; = \; \text{Arg } G_+ \; = \; \arg 1 + i \; = \; \pi/4,$$

since the angle between $1 + i$ and the real axis is 45^o, so

$$f_0 \; = \; 1/8,$$

and the period is $1/f_0 = 8$. Note that these match exactly with my "guesses" above.

4 Solution to Exercise 3.7.

Problem Statement: **(a)** *Plot the power spectrum $g(f)$ of the AR process in Exercise 3.6 and show that it has a peak at a period which is close to the period in the ACF.*

(b) *Graphically or otherwise estimate the proportion of the variance of the series in the frequency band between $f = 0.0$ and $f = 0.2$ cycle per data interval.*

Solution: The spectrum $p(f)$ for a general AR(2) process is given in (3.2.29), p. 63. The power spectrum is $g(f) = p(f)/\sigma_Z^2$. See (2.2.13), p. 40. From (3.2.28), p. 62 we have $\sigma_a^2/\sigma_Z^2 = 1 - \rho_1\phi_1 - \rho_2\phi_2 = 1 - 2/3 + 1/12 = 5/12$. For our case:

$$
\begin{aligned}
g(f) &= \frac{2 * 5/12}{1 + 1 + .25 - 2 * (1 + .5)\cos 2\pi f - 2(-.5)\cos 4\pi f} \\
&= \frac{5/6}{2.25 - 3\cos 2\pi f + \cos 4\pi f}
\end{aligned}
$$

The plot is shown below.

(fig2 goes here)

The maximum is pretty close to $f = 1/8 = .125$. It's actually about .1150 from numerical minimization (fmin in Matlab).

I'm not sure the point of part (b), but it's clear that most of the area is in the range of 0 to .2. Anyway, numerical integration using the Matlab quad function gives .8853 out of a total area of 1 (I would have guessed about 90% from looking at the graph).

Just for fun, I've given a plot of a realization of such a process below.

(fig3 goes here)

Note that it looks quasiperiodic and has about 12 periods in the record of 100 observations (as would be predicted from a period of 8).