Second Midterm Exam

Statistics 532

30 Nov. 2018

Directions

1. Please turn in the exam by 12:00 noon today.
2. Please use your own paper. **PUT YOUR NAME ON YOUR PAPER** and pledge on the front sheet. Please order all of the pages in your exam solutions and staple them.
3. This is a closed book and closed lecture notes exam. You may have one 8.5 by 11 inch sheet of paper with notes on both sides.
4. The exam is worth 100 points. The value of each question is in square brackets after the problem number. There are only 4 questions.
5. You may use results from the notes, from homeworks, or from lecture. To be on the safe side, you might give an abbreviated quotation of the result, e.g. "by the uniqueness part of the product measure theorem." Of course, you may also use results from prerequisite courses.
6. Assume all functions and sets are measurable as needed to make the mathematical expressions meaningful.

1. [30 points] Let X_1, X_2, ... be i.i.d. random variables with an exponential distribution Expo[μ] which has Lebesgue density

$$f_\mu(x) \;=\; \mu^{-1}\exp[-x/\mu]I_{(0,\infty)}(x).$$

Put $\bar{X}_n = n^{-1}\sum_{i=1}^n X_i$. You may use without proof the facts that $E_\mu[X] = \mu$ and $Var_\mu[X] = \mu^2$.

(a) Put $Y_n = 1/\bar{X}_n$ show that $\sqrt{n}(Y_n - \mu^{-1})$ converges in distribution to $N(0, \sigma^2(\mu))$ and give a formula for $\sigma^2(\mu)$.

(b) Show that

$$\frac{Y_n - 1/\mu}{\sigma(\bar{X}_n)/\sqrt{n}} \xrightarrow{D} N(0,1).$$

2. [30 points] Give the derivation of the Cramer-Rao Lower bound for variance of a statistic in a one-parameter family satisfying "regularity conditions." Indicate what regularity conditions are needed at each step of the derivation.

(3) [20 points] Let X_1, X_2, ..., X_n, Y_1, Y_2, ... Y_m be mutually independent random variables with a normal distribution with common variance σ^2. Assume $E[X_i] = \mu_X$ and $E[Y_j] = \mu_Y$. Put

$$\begin{aligned}
\bar{X} &= \frac{1}{n}\sum_{i=1}^n X_i \\
\bar{Y} &= \frac{1}{m}\sum_{j=1}^m Y_j \\
S^2 &= \frac{1}{n+m-2}\left[\sum_i (X_i - \bar{X})^2 + \sum_j (Y_j - \bar{Y})^2\right].
\end{aligned}$$

You may use without proof the fact that $(n+m-2)S^2/\sigma^2$ has a χ^2 distribution with $n+m-2$ degrees of freedom. Assume $n > 2$ and $m > 2$. Verify that the UMVUE for $(\mu_X - \mu_Y)/\sigma$ has the form

$$\frac{\bar{X} - \bar{Y}}{C_{nm}S},$$

where C_{nm} is a constant that depends only on n and m. You may invoke facts about sufficiency and completeness for the normal distribution without proving them. In particular, $(\bar{X}, \bar{Y}, S^2)$ are complete and sufficient for (μ_X, μ_Y, σ^2), and $(\bar{X}, \bar{Y})$ are complete and sufficient for (μ_X, μ_Y) assuming σ^2 is known.

4. [20 points] True or False: justify your answer to get full credit. All parts count equally.

(a) If T is a minimal sufficient statistic and a complete and sufficient statistic exists, then T is complete and sufficient.

(b) Assume X_n, $1 \leq n < \infty$ and X are real valued random variables satisfying $\forall n$, $E[|X_n|] < \infty$ and $E[|X|] < \infty$. If $X_n \stackrel{D}{\rightarrow} X$ then $E[X_n] \rightarrow E[X]$.