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Statistics 100A **Instructor: Nicolas Christou**

Examples - solutions

Example 1:

$$E(N) = E(C\pi r^2) = C\pi E r^2 = C\pi(\sigma^2 + \mu^2).$$

The mean and variance of the distribution of r are: $E(r) = 23.5$, $var(r) = 1.54$. Therefore,

$$E(N) = 8\pi(1.54 + 23.4^2) = 13800.39 \approx 13801.$$

Or simply compute $E(N)$ as:

$$E(N) = E(C\pi r^2) = C\pi \sum_r r^2 P(r) = 8\pi \left[21^2(0.05) + 22^2(0.20) + \ldots + 26^2(0.05)\right] = 13800.39 \approx 13801.$$

Example 2:
It is given that $P(X = i) = cP(X = i - 1)$ for $i = 1, 2$. Let P(X=0)=p.
$P(X = 1) = cP(X = 0) = cp$, and $P(X = 2) = cP(X = 1) = c^2p$. Also, $P(X = 0) + P(X = 1) + P(X = 2) = 1$.
Or $p + cp + c^2p = 1 \Rightarrow p = \frac{1}{1+c+c^2}$. The expected value of X is:

$$E(X) = 0(p) + 1(cp) + 2(c^2p) = \frac{c}{1 + c + c^2} + \frac{2c^2}{1 + c + c^2} = \frac{c(1 + 2c)}{1 + c + c^2}.$$

Example 3:
There are 8 white, 4 black and 2 orange balls. Two balls are selected without replacement. For each black we win \$2, for each white we lose \$1. We neither win nor we lose anything if we select an orange ball.

a.

Color	X(\$)	$P(X)$
WW	-2	$\frac{8}{14}\frac{7}{13} = \frac{56}{182}$
WO or OW	-1	$2\frac{8}{14}\frac{2}{13} = \frac{32}{182}$
OO	0	$\frac{2}{14}\frac{1}{13} = \frac{2}{182}$
BW or WB	1	$2\frac{4}{14}\frac{8}{13} = \frac{64}{182}$
BO or OB	2	$2\frac{4}{14}\frac{2}{13} = \frac{16}{182}$
BB	4	$\frac{4}{14}\frac{3}{13} = \frac{12}{182}$

b. Expected value of our winnnings:
$E(X) = -2\frac{56}{182} + -1\frac{32}{182} + 0\frac{2}{182} + 1\frac{64}{182} + 2\frac{16}{182} + 4\frac{12}{182} = 0.$

c. Standard deviation of our winnings:
$\sigma^2 = (-2)^2\frac{56}{182} + (-1)^2\frac{32}{182} + 0^2\frac{2}{182} + 1^2\frac{64}{182} + 2^2\frac{16}{182} + 4^2\frac{12}{182} - 0^2 = \frac{576}{182}$. Therefore the standard deviation is $\sigma = \sqrt{\frac{576}{182}} = 1.78.$

d. $$P(X = -2|X < 0) = \frac{P(X = -2 \cap X < 0)}{P(X < 0)} = \frac{P(X < 0|X = -2)P(X = -2)}{P(X < 0)} = \frac{1\frac{56}{182}}{\frac{56}{182} + \frac{32}{182}} = \frac{56}{88} = 0.64.$$

Example 4:
Let X be the number of tests needed for each group of 10 people. Then, if nobody has the disease 1 test is enough. But if the test is positive then there will be 11 test $(1 + 10)$. The probability distribution of X is:

X	$P(X)$
1	$\binom{10}{0}0.10^0 0.90^{10} = 0.90^{10}$
11	$1 - \binom{10}{0}0.10^0 0.90^{10} = 1 - 0.90^{10}$

Therefore the expected number of tests is:
$E(X) = 1(0.90)^{10} + 11(1 - 0.90^{10}) = 7.51.$

Example 5:
Using example 4 when $n = 2, 4, 5, 20$ we get the following: When $n = 2$, $E(X) = 1.38$. The total number of tests for the 100 people is $1.38(50) = 69$.
When $n = 4$, $E(X) = 2.38$. The total number of tests for the 100 people is $2.38(25) = 59.5$.
When $n = 5$, $E(X) = 3.05$. The total number of tests for the 100 people is $3.05(20) = 61$.
When $n = 20$, $E(X) = 18.57$. The total number of tests for the 100 people is $18.57(5) = 92.9$.
Therefore to minimize the number of tests we must place them in groups of 4.

Example 6:
The horse will win both races with probability 0.06, one race with probability 0.38, and no race with probability 0.56. The probability distribution of the profit X will be:

X	$P(X)$
80000	0.06
30000	0.38
-10000	0.56

The expected value and standard deviation of the profit are:

$$E(X) = 80000(0.06) + 30000(0.38) - 10000(0.56) = 10600.$$

$$SD(X) = \sqrt{80000^2(0.06) + 30000^2(0.38) + 10000^2(0.56) - 10600^2} = 25877.40.$$

PAGE 12 : EXAMPLES (HANDOUT #2)

EXAMPLE 3 :

$P(G) = 0.90$ $\quad P(G') = 0.10$

$P(D|G) = 0.01$ $\quad P(D|G') = 0.15$

(a). $P(D) = P(D \cap G) + P(D \cap G')$

$= P(D|G)P(G) + P(D|G')P(G')$

$= (0.01)(0.90) + (0.15)(0.10) = 0.024.$

(b). $P(G'|D) = \dfrac{P(D \cap G')}{P(D)} = \dfrac{(0.15)(0.10)}{0.024} = 0.625.$

EXAMPLE 4 :

$P(SU) = 0.40$ $\quad P(SU') = 0.60$

$P(F|SU) = 0.80$ $\quad P(F|SU') = 0.30$

(a). $P(F) = P(F \cap SU) + P(F \cap SU')$

$= P(F|SU)P(SU) + P(F|SU')P(SU')$

$= (0.80)(0.40) + (0.30)(0.60) = 0.50.$

(b). $P(SU|F) = \dfrac{P(F \cap SU)}{P(F)} = \dfrac{(0.80)(0.40)}{0.50} = 0.64.$

PAGE 15: PROBABILITY EXAMPLES (HANDOUT #2)

EXAMPLE 11:

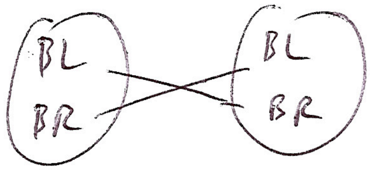
	2	2	4	4	4	6	6	10	10	10	12	12
2	4	4	6	6	6	8						
2	4	4	6	6	6	8						
4	6	6	8	8	8	10						
4	6	6	8	8	8	10						
4	6	6	8	8	8	10						
6	8	8	10	10	10	12						
6												
10												
10												
10												
12												
12												

(a). $P(\text{sum} > 6) = 1 - P(\text{sum} \leq 6) = 1 - \frac{16}{144} = \frac{128}{144} = 0.89.$

(b).

$$\left.\begin{array}{l} A\,B'\,C'\,D' \\ A'\,B\,C'\,D' \\ A'\,B'\,C\,D' \\ A'\,B'\,C'\,D \end{array}\right\} \quad \binom{4}{1} 0.89^{1}\, 0.11^{3} = 0.0047.$$

EXAMPLE 11:

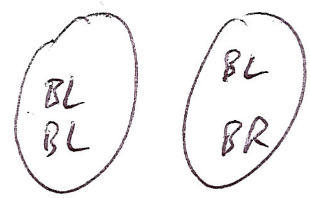
(a). 

$$P\left[\begin{array}{c}\text{A HAS ONE}\\ \text{BLUE-EYED GENE}\end{array}\;\middle|\;\begin{array}{c}\text{A HAS}\\ \text{BROWN EYES}\end{array}\right] =$$

$$\frac{\frac{1}{2}\frac{1}{2}+\frac{1}{2}\frac{1}{2}}{\frac{1}{2}\frac{1}{2}+\frac{1}{2}\frac{1}{2}+\frac{1}{2}\frac{1}{2}} = \frac{2}{3}.$$

BL, BL
BL, BR
BR, BL
BR, BR

(b). $C = \{$CHILD HAS BLUE EYES$\}$
$A = \{$A HAS ONE BLUE-EYED GENE$\}$



$$P(C \cap A) = P(C \mid A)\cdot P(A)$$
$$= \left(\tfrac{1}{2}\tfrac{1}{2}+\tfrac{1}{2}\tfrac{1}{2}\right)\tfrac{2}{3} = \tfrac{1}{3}.$$

EXAMPLE 13 :

IT IS GIVEN : $P(D) = 0.60$ $P(D') = 0.40$

$P(E|D) = 1$ $P(E|D') = 0.30$

(a). $P(E) = P(E \cap D) + P(E \cap D')$

$= P(E|D)P(D) + P(E|D')P(D')$

$= 1(0.60) + (0.30)(0.40)$

$\Rightarrow P(E) = 0.72$

(b). $P(D|E) = \dfrac{P(E \cap D)}{P(E)} = \dfrac{P(E|D)P(D)}{P(E)}$

$= \dfrac{1(0.60)}{0.72} = 0.83.$

YES, SINCE PROB. $> 80\%$.

PAGES 12 – 13 : REVIEW PROBLEMS (HANDOUT # 6)

PROBLEM 1 : START FROM RHS :

(a). $$\frac{p(n-x)}{(x+1)(1-p)} P(X=x) = \frac{p(n-x)}{(x+1)(1-p)} \binom{n}{x} p^{x} (1-p)^{n-x}$$

$$= \frac{p(n-x)}{(x+1)(1-p)} \frac{n!}{(n-x)!\,x!} p^{x} (1-p)^{n-x}$$

$$= \frac{(n-x)\,n!}{(x+1)!\,(n-x)\,(n-x-1)!} \frac{p}{1-p} p^{x} (1-p)^{n-x}$$

$$= \frac{n!}{(n-x-1)!\,(x+1)!} p^{x+1} (1-p)^{n-x-1}$$

$$= \binom{n}{x+1} p^{x+1} (1-p)^{n-x-1} = P(X=x+1).$$

(b). $$P(X=1) = \frac{0.25\,(8-0)}{(0+1)(1-0.25)} \, 0.1001 \Rightarrow P(X=1) = 0.2669.$$

$$P(X=2) = \frac{0.25\,(8-1)}{(1+1)(1-0.25)} \, 0.2669 \Rightarrow P(X=2) = 0.3114.$$

PROBLEM 6 :

(a). GEOMETRIC. $P(X=1) = 0.4$

$P(X=2) = (0.6)(0.4) = 0.24$

$P(X=3) = (0.6^2)(0.4) = 0.144$

(b). NEGATIVE BINOMIAL.

$$P(X=4) = \binom{4-1}{2-1} 0.4^2 \, 0.6^2 = 0.1728.$$

PROBLEM 7 : $X \sim \text{Poisson}(2)$

$EX = \mu = 2$

$VAR(X) \equiv \sigma^2 = 2$

$Y = 50 - 2X - X^2$

$$EY = E(50 - 2X - X^2)$$
$$= 50 - 2EX - EX^2$$
$$= 50 - 2\mu - (\sigma^2 + \mu^2)$$
$$= 50 - 2(2) - (2 + 2^2) \Rightarrow \underline{EY = 40}.$$

PROBLEM 8:

$$P(Y=x) = P(X=x \mid X>0) = \frac{P(X=x \cap X>0)}{P(X>0)}$$

$$= \frac{P(X>0 \mid X=x)\,P(X=x)}{P(X>0)} = \frac{P(X=x)}{P(X>0)}$$

$$\therefore\ P(Y=x) = \frac{P(X=x)}{P(X>0)}$$

$$\left.\begin{aligned} P(Y=1) &= \frac{P(X=1)}{P(X>0)} \\ P(Y=2) &= \frac{P(X=2)}{P(X>0)} \end{aligned}\right\} \Longrightarrow$$

Y	$P(y)$
1	$\frac{P(X=1)}{P(X>0)}$
2	$\frac{P(X=2)}{P(X>0)}$
3	$\frac{P(X=3)}{P(X>0)}$
$\vdots$	$\vdots$

THEREFORE, $$EY = \sum_{y=1}^{\infty} y\,P(y)$$

$$= \frac{\sum_{x=0}^{\infty} x\,P(X=x)}{P(X>0)} = \frac{EX}{1-P(X=0)}$$

$$= \frac{\lambda}{1-\frac{\lambda^0 e^{-\lambda}}{0!}} \Rightarrow \boxed{EY = \frac{\lambda}{1-e^{-\lambda}}}$$