Chapter 12
Integral Calculus

Newton

Leibniz

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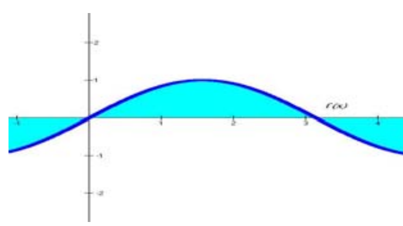
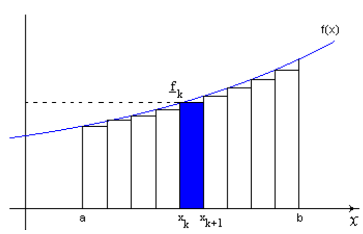
12. Problem: Area under a curve

- A very old problem (Archimedes proposed a solution! Fermat worked on it, too). Newton and Leibniz solved it in late 17^{th} century .



- Idea: Introduce rectangles under the curve, defined by *f(x)*, find the area of all of those rectangles and add them all up.



- Rigorous mathematical details had to wait till the 19^{th} century.

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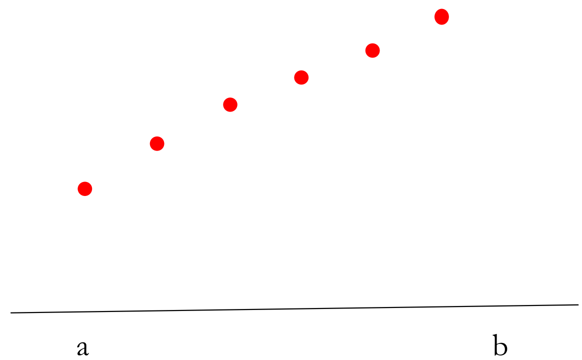
12. Estimating Area Under Points

• What if instead of a function, we were given points: how could we use rectangles to estimate area under points?



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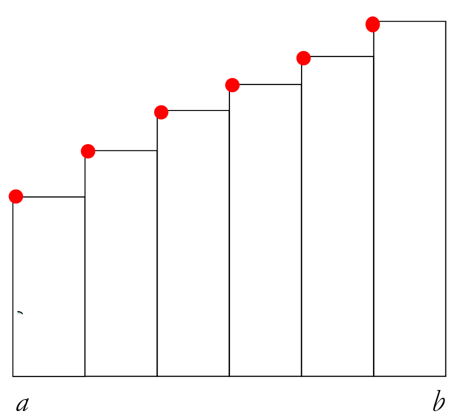
12. Underestimating Area

• Here we underestimate the area by putting left corners at points:



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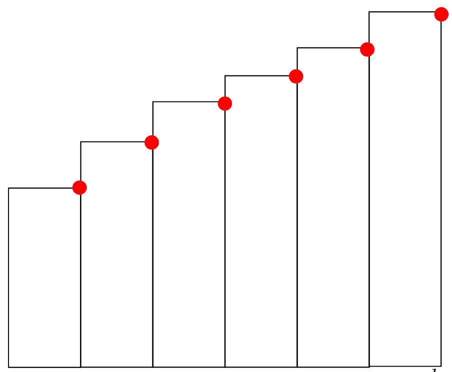
12. Overestimating Area

- Here we overestimate the area by putting right corners at points:



• Note: As the intervals become smaller and smaller –i.e., the partitions of [*a,b*] become finer-, both the left corner and the right corner based areas converge.

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12.1 Riemann Sum

• In general, integration is motivated as an area under a curve. This geometric intuition is fine.

• But, we want to think of *integration* as a form of summation. In economics and finance, geometric interpretations, in general, have little use. But, the summation intuition works very well.

• Riemann thought of an integral as the convergence of two sums, as the partition of the interval of integration becomes smaller.

Bernhard Riemann (1826-1866), Germany

12.1 Riemann Sum

• In order to estimate an area, we need a partition of the interval $[a,b]$. We define a *partition* P of the closed interval $[a,b]$ as a finite set of points $P = \{ x_0, x_1, x_2, ..., x_n\}$ such that

$$a = x_0 < x_1 < x_2 < ... < x_{n-1} < x_n = b.$$

• If $P = \{ x_0, x_1, x_2, ..., x_n\}$ is a partition of the closed interval $[a,b]$ and f is a function defined on that interval, then the *n-th Riemann Sum of f with respect to the partition P* is defined as:

$$R(f, P) = \Sigma_{j=1 \text{ to } n} f(t_j)\,(x_j - x_{j-1})$$

where t_j is an arbitrary number in the interval $[x_{j-1}, x_j]$.

• But, we do not know t_j. In the previous two examples, we used the left end points of the interval $[x_{j-1}, x_j]$ (underestimation of area) and the right points of the interval $[x_{j-1}, x_j]$ (overestimation of area).

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12.1 Riemann Sum

• There are two useful cases:

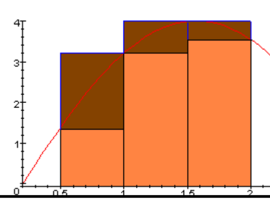
1) use c_j, the supremum of $f(x)$ in the interval $[x_{j-1}, x_j]$, producing the *upper sum*:

$$U(f, P) = \Sigma_{j=1 \text{ to } n}\, c_j\,(x_j - x_{j-1})$$

2) use d_j, the infimum of $f(x)$ in the interval $[x_{j-1}, x_j]$, producing the *lower sum*:

$$L(f, P) = \Sigma_{j=1 \text{ to } n}\, d_j\,(x_j - x_{j-1})$$

<u>Example</u>: $U(f, P)$ is displayed in dark brown and $L(f, P)$ in orange.



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12.1 Riemann Sum

Proposition: Size of Riemann Sums

Let P be a partition of the closed interval $[a,b]$, and $f(.)$ be a bounded function defined on that interval. Then,

- The lower (upper) sum is increasing (decreasing) with respect to refinements of partitions --i.e. $L(f, P') \geq L(f, P)$ or $U(f, P') \leq U(f, P)$ for every refinement P' of the partition P.

- $L(f, P) \leq R(f, P) \leq U(f, P)$ for every partition P

That is, the lower sum is always less than or equal to the upper sum.

Q: Will $U(f, P)$ and $L(f, P)$ ever be the same?

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12.1 Riemann Integral

• Suppose $f(.)$ is a bounded function defined on a closed, bounded interval $[a, b]$. Define the *upper* and *lower Riemann integrals* as:

$I^*(f) = inf \{U(f,P)$: P a partition of $[a, b]\}$
$I_*(f) = sup \{L(f,P)$: P a partition of $[a, b]\}$

Then if $I^*(f) = I_*(f)$ the function $f(.)$ is called *Riemann integrable* (*R-integrable)* and the *Riemann integral* of $f(.)$ over the interval $[a, b]$ is denoted by

$$\int_a^b f(x)dx$$

Note: $U(f, P)$ and $L(f, P)$ depend on the chosen partition, while the upper and lower integrals are independent of partitions. But, this definition is not practical, since we need to find the *sup* and *inf* over *any* partition.

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12.1 Riemann Integral

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12.1 Riemann Integral – Example 1

Example: Is $f(x)=x^2$ R-integrable on [0,1]?

It is complicated to prove that this function is integrable. We do not have a simple condition to tell us whether this, or any other function, is integrable.

But, we should be able to generalize the proof for this particular example to a wider set of functions.

First, note that in the definition of upper and lower integral it is not necessary to take the *sup* and *inf* over all partitions: If P is a partition and P' is a refinement of P, then

$$L(f, P') \geq L(f, P) \quad \text{and} \quad U(f, P') \leq U(f, P).$$

Thus, partitions with large intervals (large *norms,* $|P|$) do not contribute to the *sup* or *inf*. We look at partitions with a small norms.

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12.1 Riemann Integral – Example 1

Second, take any $\varepsilon > 0$ and a partition P with $|P| < \varepsilon/2$. Then,

$$|U(f, P) - L(f, P)| \leq \Sigma_{j=1 \text{ to } n} |c_j - d_j| (x_j - x_{j-1}),$$

where c_j is the *sup* of f over $[x_{j-1}, x_j]$ and d_j is the *inf* over that interval.

Since $f(.)$ is increasing over [0, 1], we know that the *sup* is achieved on the right side of each subinterval, the *inf* on the left side. Then,

$$|U(f, P) - L(f, P)| \leq \Sigma_{j=1 \text{ to } n} |c_j - d_j| (x_j - x_{j-1})$$
$$= \Sigma_{j=1 \text{ to } n} |f(x_j) - f(x_{j-1})| (x_j - x_{j-1})$$

To estimate this sum, we use the Mean Value Theorem for $f(x) = x^2$:

$$|f(x) - f(y)| \leq |f'(c)| |x - y| \text{ for } c \text{ between } x \text{ and } y.$$

Since $|f'(c)| \leq 2$ for $c \in [0, 1]$ => $|f(x) - f(y)| \leq 2 |x - y|$

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12.1 Riemann Integral – Example 1

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But P was chosen with $|P| < \varepsilon/2$

$$\Rightarrow |f(x_j) - f(x_{j-1})| \leq 2 |x_j - x_{j-1}| \leq 2 \varepsilon / 2 = \varepsilon.$$

Then, $|U(f, P) - L(f, P)| \leq \Sigma_{j=1 \text{ to } n} |f(x_j) - f(x_{j-1})| (x_j - x_{j-1})$

$$\leq \varepsilon \Sigma_{j=1 \text{ to } n} (x_j - x_{j-1}) = \varepsilon (1 - 0) = \varepsilon.$$

Since P was arbitrary but with small norm --sufficient for the upper and lower integrals--, the upper and lower integral must exist and be equal to one common limit L.

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12.1 Riemann Integral – Example 1

• Let's calculate L. It is easy, since now we know that the function is integrable. Then, we take a suitable partition to find the value of the integral. For example, take the following partition

$x_j = j/n$ for $j = 0, 1, 2, ..., n$.

Then, the upper sum:

$$U(f, P) = \Sigma_{j=1 \text{ to } n}\, c_j\, (x_j - x_{j-1}) = \Sigma_{j=1 \text{ to } n}\, f(x_j)\, 1/n$$
$$= \Sigma_{j=1 \text{ to } n}\, (j/n)^2\, 1/n = 1/n^3\, \Sigma_{j=1 \text{ to } n}\, j^2$$
$$= 1/n^3\, [1/6\, n\, (n+1)\, (2n+1)] = 1/6\, (n+1)\, (2n+1)/n^2$$

Since we know that the upper integral exists and is equal to L, the limit as n goes to infinity of the above expression must also converge to L. Then, $L = 1/3$.

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12.1 Riemann Integral – Example 2

Example: Is the Dirichlet function R-integrable?

The Dirichlet Function (Q is the set of rational numbers):

$$f(x) = \begin{cases} 1 & \text{if } x \in Q \\ 0 & \text{if } x \notin Q \end{cases}$$

We have that $U(f, P) = 1$ and $L(f, P) = 0$, regardless of P. Then,

$I^*(f) = 1$ and $I_*(f) = 0$.

Thus, the Dirichlet function is *not R-integrable* over the interval $[a,b]$.

Note: Unlike the function in the previous example, we have a discontinuous function on the Irrational Numbers. We have infinite discontinuities.

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12.1 Riemann Lemma

• The first example shows that it is difficult to establish the integrability of a given function. The second example illustrates that not every function is Riemann integrable.

• The *Rienmann lemma* provides an easier condition to check the integrability of a function.

Suppose *f(.)* is a bounded function defined on the closed, bounded interval $[a, b]$. Then, *f(.)* is R-integrable if and only if for every $\varepsilon>0$ there exists at least one partition P such that

$$|U(f,P) - L(f,P)| < \varepsilon$$

In example 1, we check the above inequality holds for *every* partition P with small enough norm. Using Riemann's Lemma, we only need to check the inequality holds for one partition. Easier! 17

12.1 Riemann Integral - Remarks

• Roughly speaking, we define the Riemann integral as follows:
- Subdivide the domain of the function (usually a closed, bounded interval) into finitely many subintervals (the partition)
- Construct a simple function that has a constant value on each of the subintervals of the partition (the Upper and Lower sums)
- Take the limit of these simple functions as you add more and more points to the partition.

• If the limit exists, it is called the Riemann integral and the function is called R-integrable.

• A function is R-integrable on $[a,b]$ if:
  - It is continuous on $[a,b]$
  - It is monotone on $[a,b]$
  - It is bounded, with a finite number of discontinuities on $[a,b]$.

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12.1 Riemann Integral - Remarks

• For a function to be R-integrable it must be bounded. If the function is unbounded even at a single point in an interval [a, b] it is not Riemann integrable (because the *sup* or *inf* over the subinterval that includes the unbounded value is infinite). For example, $f(x)= 1/x$ over [0,1], unbounded at x=0.

• The Riemann integral is based on the concept of an "interval", or rather on the length of subintervals [x_{j-1}, x_j]. The concept of partition applies to an interval. We can take Riemann integrals over unions of intervals, but nothing more complicated (say, Cantor sets).

• Partitions depend on the structure of the real line. Thus, we cannot define a R-integrable for functions defined on more abstract spaces --say, sequences, functions from N to R.

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12.1 Riemann-Stieljes Integral

• The Riemann–Stieltjes integral of a real-valued function f of a real variable with respect to a real function g is denoted by:

$$\int_a^b f(x)\ dg(x)$$

defined to be the limit, as the mesh of the partition

$$P = \{a = x_0 < x_1 < x_2 < ... < x_{n-1} < x_n = b\},$$

of the interval [a, b] approaches zero, of the approximating sum

$$S(f,g,P) = \Sigma_{i=1 \text{ to } n} f(c_i)\ (g(x_j) - g(x_{j-1})),$$

where c_i is in the i-th subinterval [x_{i-1}, x_i]. The two functions f and g are respectively called the integrand and the integrator.

• If g is everywhere differentiable, then the Riemann–Stieltjes integral may be different from the Riemann integral of $f(x)\,g'(x)$. For example, if the derivative is unbounded. But if the derivative is continuous, they will be the same.

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12.1 Lebesgue's Theory - Introduction

• The Riemann integral is based on partitioning the domain $[a,b]$ in subintervals $[x_{j-1}, x_j]$, picking a point x_j^* in the subinterval and calculating the area under the curve by computing the Riemann sums. Then, take the limit as we add more and more points to the partition.

• Roughly speaking, Lebesgue's theory, instead of partitioning the domain, partitions the *range* into subintervals.

• Based on these subintervals, calculate areas and sum over these areas. The approximation improves with finer and finer partitions of the range.

• The Lebesgue integral will be the limit of these sums.

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12.1 Lebesgue's Theory - Introduction

• Suppose the function takes values between $[c,d]$.

1) Divide range $[c,d]$ into subintervals: $[c=y_0,y_1]$, $[y_1,y_2]$, ... ,$[y_{N-1},y_N=d]$

2) Define E_i as the set of all points in $[a,b]$ whose value under f lies between y_i and y_{i+1}: $E_i = f^{-1}([y_i,y_{i+1}]) = \{x \in [a,b] \mid y_i \leq f(x) \leq y_{i+1}\}$.

3) Assign a "size" to E_i -a *measure* $\mu(E_i)$. Then, the portion of the graph of $y=f(x)$ between the horizontal lines $y=y_i$ & $y= y_{i+1}$ will be A_i, where, $y_i\ \mu(E_i) \leq A_i \leq y_{i+1}\ \mu(E_i)$.

4) Approximate the area by picking a number $y_i^* \in [y_i,y_{i+1}]$, and compute: $\sum_{i=0}^{n-1} y_i^*\ \mu(E_i)$

5) The approximation improves with finer and finer partitions of $[c,d]$. The Lebesgue integral will be the limit (if it exists) of these sums. The function is called *Lebesgue integrable.*

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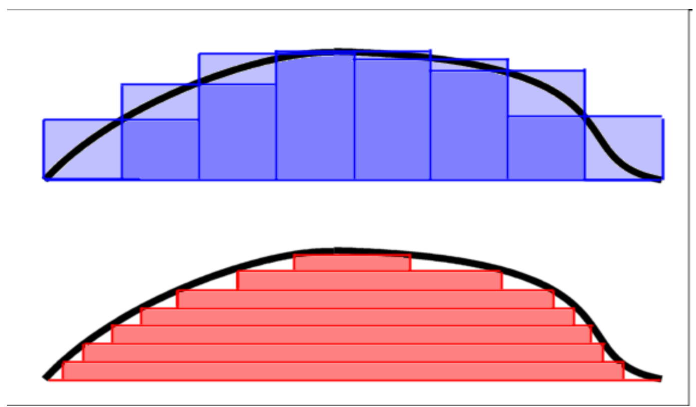
12.1 Lebesgue's Theory - Introduction

- Riemann & Lebesgue integrals: Vertical sums vs Horizontal sums.



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12.1 Lebesgue's Theory - Introduction

• Riemann & Lebesgue integrals: Analogy:
You have a pile of coins and you want to know how much money you have. For this purpose, you can pick the coins randomly, one by one, and add them. This is the Riemann integral.
You can also sort the coins by denomination first, and get the total by multiplying each denomination by how many you have of that denomination and add them up. This is the Lebesgue integral.

• The methods are different, but you obtain the same result by either method. Similarly, when both the Riemann integral and the Lebesgue integral are defined, they give the same value.

• But, there are functions for which the Lebesgue integral is defined but the Riemann integral is not. In this sense, the Lebesgue integral is more general than the Riemann.

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12.1 Lebesgue's Theory - Measure

• Step 3 assigns a size to the set E_i. With Riemann integrals, we use as size the length of subintervals $[x_{j-1}, x_j]$. This is fine, but we need to generalize the concept to more complicated sets. The generalization, called *measure*, is a function assigning to each set $\boldsymbol{A}$ in R^m a non-negative number, $\mu(\boldsymbol{A})$.

• This *measure* should satisfy two conditions:
1) It should be applicable to intervals, unions of intervals, and to more general sets (say, a Cantor set). Ideally, defined for all sets.
2) It should share many of the properties of "length of an interval":
- $\mu(\boldsymbol{A}) \geq 0$ (Non-negative);
- $\mu(\boldsymbol{A} = [a,b]) = b - a$;
- Invariant under translation: if $F=E+c=\{e+c \mid e \in E\} \Rightarrow \mu(F)=\mu(E)$;
- Countably additive, i.e., $\mu(\boldsymbol{A} = \cup A_n) = \Sigma_n (A_n)$, where A_n are pairwise disjoint sets. 25

12.1 Lebesgue's Theory – Lebesgue Measure

• Lebesgue defined a measure, the *Lebesgue measure*, that satisfies both conditions:
- First, define an outer measure (based on the infimum of a set), which satisfies (1):

If $\boldsymbol{A}$ is any subset of $\boldsymbol{R}$, define the (*Lebesgue*) *outer measure* of $\boldsymbol{A}$ as:

$$\lambda^*(\boldsymbol{A}) = inf\ \{\Sigma\, l(\boldsymbol{A}_n)\}$$

where the infimum is taken over all countable collections of open intervals $\boldsymbol{A}_n$ such that $\boldsymbol{A} \subset \cup \boldsymbol{A}_n$ and $l(\boldsymbol{A}_n)$ is the standard length of the interval $\boldsymbol{A}_n$.

Note: λ^* is defined for all sets, but, λ^* is not additive, it is subadditive –i.e., $\lambda^*(F \cup E) \leq \lambda^*(F) + \lambda^*(E)$; => not quite length.

• The outer measure is a real-valued, non-negative, monotone and countably subadditive set function. 26

12.1 Lebesgue's Theory – Measurable Sets

- Second, define *measure* by restricting the outer measure to measurable sets:

A set $\boldsymbol{E}$ is (*Lebesgue*) *measurable* if for every set $\boldsymbol{A}$ we have that

$$\lambda^*(\boldsymbol{A}) = \lambda^*(\boldsymbol{A} \cap \boldsymbol{E}) + \lambda^*(\boldsymbol{A} \cap \boldsymbol{E}^C)$$

If $\boldsymbol{E}$ is measurable, the non-negative number $\mu(\boldsymbol{E}) = \lambda^*(\boldsymbol{E})$ is the (Lebesgue) measure of the set $\boldsymbol{E}$.

Example 1: The set R of all real numbers is measurable:

$\lambda^*(\boldsymbol{A} \cap R) + \lambda^*(\boldsymbol{A} \cap R^C) = \lambda^*(\boldsymbol{A}) + \lambda^*(\boldsymbol{A} \cap \emptyset) = \lambda^*(\boldsymbol{A})$

=> R is measurable.

Example 2: The complement of a measurable set is measurable.

Suppose $\boldsymbol{E}$ is *measurable* => $\lambda^*(\boldsymbol{A}) = \lambda^*(\boldsymbol{A} \cap \boldsymbol{E}) + \lambda^*(\boldsymbol{A} \cap \boldsymbol{E}^C)$

For $\boldsymbol{E}^C$ we have:

$$\lambda^*(\boldsymbol{A} \cap \boldsymbol{E}^C) + \lambda^*(\boldsymbol{A} \cap (\boldsymbol{E}^C)^C) = \lambda^*(\boldsymbol{A} \cap \boldsymbol{E}^C) + \lambda^*(\boldsymbol{A} \cap \boldsymbol{E}) = \lambda^*(\boldsymbol{A})$$

=> $\boldsymbol{E}$ is measurable.

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12.1 Lebesgue's Theory – Measurable Sets

• This restriction makes the measure additive, satisfying (2), mainly the additive requirement. But, now the measure is not defined for all sets, since not all sets are measurable (the *axiom of choice* in set theory plays a role here).

• Lebesgue's definition of measurable sets is not very intuitive. But, it is elegant, general, brief and it works.

Remark: Not *every* set is measurable, but it is fair to say that *most* sets are.

• Usually, the family of all measurable sets is denoted by $\mathscr{M}$ (script M). $\mathscr{M}$ is a sigma-algebra and translation invariant set containing all intervals.

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12.1 Lebesgue's Theory –Properties of Measure

• Properties of Lebesgue measure:
1. All intervals are measurable. The measure of an interval: its length.
2. All open and closed sets are measurable.
3. The union and intersection of a finite or countable number of measurable sets is again measurable.
4. If $\boldsymbol{A}$ is measurable and $\boldsymbol{A}$ is the union of a countable number of measurable sets $\boldsymbol{A}_n$, then $\mu(\boldsymbol{A}) \leq \Sigma\, \mu(\boldsymbol{A}_n)$.
5. If $\boldsymbol{A}$ is measurable and $\boldsymbol{A}$ is the union of a countable number of disjoint measurable sets $\boldsymbol{A}_n$, then $\mu(\boldsymbol{A}) = \Sigma\, \mu(\boldsymbol{A}_n)$.

• According to these properties, most common sets are measurable: intervals; closed & open sets; unions & intersections of measurable sets. But, the property that measure is (countably) additive implies that not *every* set is measurable.

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12.1 Lebesgue Integral – Measurable functions

• In Lebesgue's theory, integrals are defined for a class of functions called *measurable functions,* which are the ones for which the sets we get are measurable sets.

• A function f: A →[-∞,∞] is *measurable* (or measurable on A) if A ∈$\mathscr{M}$ and the pre-image of every interval of the form (t, ∞) is in $\mathscr{M}$:

$$\{x \,|\, f(x)>t\} \in \mathscr{M} \text{ for all } t \in R$$

This is somewhat comparable to one of the definitions of continuous functions: A function f is continuous if the inverse image of every open interval is open. However, not every measurable function is continuous, while every continuous function is clearly measurable.

<u>Note</u>: Simple functions, step functions, continuous functions, and monotonic functions are measurable.

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12.1 Lebesgue Integral

• **Proposition**. Let f & g be measurable extended real-valued functions on A ∈ $\mathscr{M}$. Then, the following functions are all measurable on A:

$f + c; cf\ ; f + g; f g,$ where $c \in R$.

<u>Note</u>: Usual convention to avoid nonsense results, when f (and/or g)= ∞/- ∞ : 0 x ∞ = 0.

• Now, we have the tools to define, and possibly compute, the integral for those functions:

$$\int_A f(x)\ \lambda(dx)$$

that represents, loosely, a limit of integral sums

$$\Sigma_i f(x_i)\ \lambda(A_i),$$

where λ is a measure on some space $\boldsymbol{A}$, and $\{A_i\}$ is a partition of $\boldsymbol{A}$, and x_i is a point in A_i .

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12.1 Lebesgue Integral

• To define the Lebesgue integral, we usually follow these steps:
- Define the Lebesgue integral for "simple functions."
- Define the Lebesgue integral for bounded functions over sets of finite measure.
- Extend the Lebesgue integral to positive functions (not necessarily bounded). (The concept of measurable function plays a role.)
- Define the general Lebesgue integral.

• **Definition**: Simple function

$$\varphi(\omega) = \Sigma_{i=1 \text{ to } n}\ a_i\ I_{Ai}(\omega),$$

where $A_1, ..., A_k$ are measurable sets on Ω, I_{Ai} is an indicator function and $a_1, ..., a_k$ are real numbers. Let $A_1, ..., A_k$ be a partition of Ω --i.e., A_i's are disjoint and $A_1 \cup ... \cup A_k = \Omega$.

Then, $\varphi(.)$ with distinct a_i's exactly characterizes this partition.

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12.1 Lebesgue Integral – Simple functions

• Simple functions can be thought of as dividing the *range* of f, where the resulting sets $\boldsymbol{A}_n$ may or may not be intervals.

Example: A step function, $\varphi(\omega) = a_i$ for $x_{j-1} < x < x_j$ and the $\{x_j\}$ form a partition of $[a, b]$. Upper, Lower, and Riemann sums are examples of step functions.

• **Definition**: Lebesgue Integral for Simple Functions
Let $\varphi(x) = \Sigma_n a_n I_{An}(x)$ be a simple function and $\mu(\boldsymbol{A}_n)$ be finite for all n, then the *Lebesgue integral of* φ is defined as:

$$\int \varphi(x)\, dx = a_1\, \mu(\boldsymbol{A}_1) + a_2\, \mu(\boldsymbol{A}_2) \ldots + a_n\, \mu(\boldsymbol{A}_n) = \Sigma_n a_n\, \mu(\boldsymbol{A}_n)$$

If $\boldsymbol{E}$ is a measurable set, we define

$$\int_E \varphi(x)\, dx = \int I_E(x)\, \varphi(x)\, dx$$

Recall: $\mu(\boldsymbol{A} = [a,b]) = b - a$. Thus, $\mu(x_j, x_j + dx) = dx$

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12.1 Lebesgue Integral – Simple functions

Example 1: Lebesgue integral of a step function $f(x)$, defined as:

$$f(x) = \begin{cases} 1 \text{ if } -1 < x < 2 \\ 2 \text{ if } 2 \leq x < 4 \\ 3 \text{ if } 4 \leq x \leq 8 \\ 0 \text{ otherwise} \end{cases}$$

$$\begin{aligned} \int f(x)\, dx &= a_1\, \mu(\boldsymbol{A}_1) + a_2\, \mu(\boldsymbol{A}_2) \ldots + a_n\, \mu(\boldsymbol{A}_n) \\ &= 1\, \mu([-1,2]) + 2\, \mu([2,4]) + 3\, \mu([4,8]) \\ &= 1\text{x}3+2\text{x}2+3\text{x}4 = 19 \end{aligned}$$

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12.1 Lebesgue Integral – Simple functions

Example 2: Lebesgue integral of $f(x) = c$
$f(x) = c$ can be written as a simple function $f(x) = c\, I_R(x)$
Then, the Lebesgue integral of f over $[a,b]$ is by definition:

$$\int_{[a,\, b]} f(x)\, dx = \int I_{[a,\, b]}(x)\, f(x)\, dx =$$
$$= \int c\, I_{[a,\, b]}(x)\, dx = c\, \mu([a,\, b]) = c\,(b - a)$$

Note: The same answer as for the Riemann integral.

• Example 3: Lebesgue integral of Dirichlet function on [0,1].
Let **Q** be the set of all rational numbers, then the Dirichlet function restricted to [0, 1] is the indicator function of **A** = **Q** ∩ [0, 1]. The set **A** is a subset of **Q**, hence **A** is measurable and μ(**A**) = 0.
Thus,

$$\int I_A(x)\, dx = \mu(A) = 0$$

Note: Not the same answer as in Riemann's case. 35

12.1 Lebesgue Integral – Bounded functions

• We used step functions to define the R-integral of a bounded function f over an interval $[a,b]$. Now, we use simple functions to define the L-integral of f over a set of finite measure.

• **Definition**: Lebesgue Integral for Bounded Functions
Suppose f is a bounded function defined on a measurable set $\boldsymbol{E}$ with finite measure. Define the *upper* and *lower* Lebesgue integrals as

$$I^*(f)_L = \inf\{\varphi(x)\, dx\text{: } \varphi \text{ is simple and } \varphi \geq f\ \} \qquad \text{(lower)}$$
$$I_*(f)_L = \sup\{\varphi(x)\, dx\text{: } \varphi \text{ is simple and } \varphi \leq f\ \} \qquad \text{(upper)}$$

If $I^*(f)_L = I_*(f)_L$ the function f is called *Lebesgue integrable* (*L-integrable*) over $\boldsymbol{E}$ and the Lebesgue integral of f over $\boldsymbol{E}$ is denoted by

$$\int_E f(x)\, dx$$

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12.1 Lebesgue Integral – Bounded functions

Example: Is $f(x)=x$ L-integrable over [0,1]?
We know that $|f(x)| \leq 1$ over the interval [0,1]. Define sets:

$E_j = \{x \in [0,1]: (j-1)/n \leq f(x) < j/n\}$ for $j = 1, 2, ..., n$.

Because f is continuous, the sets E_j are measurable, they are disjoint, and their union (over the j's) equals [0,1].

Define two simple functions

$S_n(x) = \Sigma_j \; j/n \, I_{Ej}(x)$
$s_n(x) = \Sigma_j \; (j-1)/n \, I_{Ej}(x)$

Fix an integer n and take a number $x \in [0,1)$. Then, x must be contained in exactly one set E_j, and on that set we have

$s_n(x) = (j-1)/n \leq f(x) < j/n = S_n(x)$

Thus, on all of [0,1], we know that $s_n(x) \leq f(x) \leq S_n(x)$

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12.1 Lebesgue Integral – Bounded functions

Example (continuation): Thus, on all of [0,1], we know that $s_n(x) \leq f(x) \leq S_n(x)$.
But then,

$I^*(f)_L \leq \int S_n(x)\, dx = 1/n \; \Sigma_j \; j \; \mu(E_j)$
$I_*(f)_L \geq \int s_n(x)\, dx = 1/n \; \Sigma_j \; (j-1) \; \mu(E_j)$

Therefore,

$I^*(f)_L - I_*(f)_L \leq 1/n \; \Sigma_j \; (j - (j-1)) \; \mu(E_j) = 1/n \; \Sigma_j \; \mu(E_j)$
$= 1/n \; \mu([0,1]) = 1/n$

Since n was arbitrary the upper and lower Lebesgue integrals must agree, hence the function f is L-integrable.

Note: With a few simple modifications this example can be used to show that *every bounded function f,* which has the property that *the sets* $\boldsymbol{E_j}$ *are measurable,* is L-integrable.

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12.1 Lebesgue Integral – Bounded functions

Example: Value of the Lebesgue integral $f(x) = x$ over [0,1]
Compute $\mu(E_j)$ using the fact that $f(x) = x$: for a fixed n we have

$\mu(E_j) = \mu(\{x \in [0,1]: (j-1)/n < f(x) < j/n\}) =$
$= \mu(\{x \in [0,1]: (j-1)/n < x < j/n\}) =$
$= \mu([(j-1)/n, j/n]) = 1/n$

Then,

$\int f(x)\, dx = \lim 1/n\, \Sigma_j\, j\, \mu(E_j)$
$= \lim 1/n\, \Sigma_j\, j\, 1/n$
$= \lim 1/n^2\, \Sigma_j\, j = \lim 1/n^2\, [n(n-1)/2] = 1/2$

=> same value as for the Riemann integral.

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12.1 Lebesgue Integral – General Case

• We have extended the concept of integration to (bounded) functions defined on general sets (measurable sets with finite measure) without using partitions (subintervals).

• The Lebesgue integral agrees with the Riemann integral, when both apply. This new concept removes some strange results –for example, we can integrate over Dirilecht functions over an interval.

• But, we have restricted our attention to *bounded* functions only. To generalize the Lebesgue integral to functions that are unbounded, including functions that may occasionally be equal to infinity, we need the concept of a *measurable function*.

• Recall that measurable functions do not have to be continuous. They may be unbounded and they can be equal to $\pm\infty$. They are "almost" continuous –i.e., except on a set of measure less than ε.

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12.1 Lebesgue Integral – General Case

Definition: Lebesgue Integral of Non-Negative Functions
Let f be a measurable function defined on E and h be a bounded measurable function such that $\lambda(\{x: h(x) > 0\})$ is finite, then we define

$$\int_E f(x)\,dx = sup\{\int_E h(x)\,dx,\ h \leq f\}$$

If $\int_E f(x)\,dx$ is finite, then f is called L-integrable over E.

Definition: General Lebesgue Integral
Let f be a measurable function. Define the positive and negative parts of f, respectively, as:

$$f^+(x) = max(f(x), 0)$$
$$f^-(x) = max(-f(x), 0)$$

so that $f = f^+ - f^-$. Then, f is Lebesgue integrable if f^+ and f^- are L-integrable and

$$\int_E f(x)\,dx = \int_E f^+(x)\,dx - \int_E f^-(x)\,dx$$

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12.1 Lebesgue Integral - Remarks

• The Lebesgue integral is more general than the Riemann integral: If $f(.)$ is R-integrable, it is also L-integrable.

• For most practical applications, we use the result that for continuous functions or bounded functions with at most countably many discontinuities over intervals $[a,b]$ there is no need to distinguish between the Lebesgue or Riemann integral.

• Then, all Riemann integration techniques can be used. But, for more complicated situations, the Lebesgue integral is more useful.

• The Lebesgue integral makes no distinction between bounded and unbounded sets in integration, and the standard theorems apply equally to both cases.

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12.1 Lebesgue Integral - Remarks

• For theoretical purposes the Lebesgue integral provides an abstraction level that simplifies proofs.

• But, then techniques such as integration by parts or substitution may no longer apply.

• It plays a pivotal role in the axiomatic theory of probability. A probability measure behaves analogously to an area measure, and, in fact, a probability measure is a measure in the Lebesgue sense.

• There are several other generalizations of the Riemann integral: Perron, Denjoy, Henstock, etc.

H. Lebesgue (1875-1941, France)

12.1 Notation

- *f*(*x*): function (it must be continuous in *[a,b]*).
- *x*: variable of integration
- *f*(*x*) d*x*: integrand
- *a, b*: boundaries

$$\int_a^b f(x)\,dx$$

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12.1 Properties of Integrals

Assuming $f(x)$ and $g(x)$ are Riemann integrable functions on $[a,b]$, with c inside $[a,b]$ and k and q are constants, the following properties can be derived (the last three are easy if we think of Riemann integration as summation):

$$\int_a^a f(x)dx = 0$$

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

$$\int_a^b [kf(x) + qg(x)]dx = k\int_a^b f(x)dx + q\int_a^b g(x)dx$$

$$\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|\,dx$$

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12.2 Fundamental Theorem of Calculus

- The fundamental theorem of calculus states that differentiation and integration are inverse operations.
- It relates the values of antiderivatives to definite integrals. Because it is usually easier to compute an antiderivative than to apply the definition of a definite integral, the Fundamental Theorem of Calculus provides a practical way of computing definite integrals.
- It can also be interpreted as a precise statement of the fact that differentiation is the inverse of integration.

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12.2 Fundamental Theorem of Calculus

- The Fundamental Theorem of Calculus states: If a function *f* is continuous on the interval [*a*, *b*] and if *F* is a function whose derivative is *f* on the interval (*a*, *b*), then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

- Furthermore, for every *x* in the interval (*a*, *b*),

$$F(x) = \int_a^x f(t)dt, \quad \text{satisfying } \frac{dF(x)}{dx} = f(x)$$

- In other words, if a function has a derivative over a range of numbers, the integral over that same range can be calculated by evaluating at the end points of the range and subtracting.

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12.2 Fundamental Theorem of Calculus: Notes

- The first part is used to evaluate integrals.
- The second part defines the anti-derivative. Finding the anti-derivative is finding the integral.

- <u>Example</u>: Find the antiderivative of $f(x) = 10\ x^4$
  $F(x) = 2\ x^5$

- In general, small letters will be used for functions, capital letters for anti-derivatives.

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12.2 Physics Example: Velocity & Acceleration

- Velocity, v(t), is defined as the derivative of position, x(t). Acceleration, a(t), is the derivative of v(t), and the second derivative of x(t).
- The integral of acceleration is velocity. The integral of velocity is position.
- The graph of v(t) against time, t, shows that the area between two times is the distance traveled.
- On the other hand, the area under a(t) against time shows the velocity.

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12.3 Rules of Integration

Integration of the power function:

For $n \neq -1$ $\int \kappa x^n dx = \frac{\kappa}{n+1} x^{n+1} + C$

Integration of $\frac{1}{x}$ $\int \frac{\kappa}{x} dx = \kappa \ln(x) + C$

The Integral of e^x $\int e^{\lambda x} dx = \frac{e^{\lambda x}}{\lambda} + C$

The Constant Multiple Rule $\int cf(x)dx = c\int f(x)dx$

The Sum Rule for Integrals $\int [f(x)+g(x)]dx = \int f(x)dx + \int g(x)dx$

Integration of sin function: $\int \sin \alpha x dx = \frac{-\cos \alpha x}{\alpha} + C$

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12.3 Rules of Integration: Applied Joke

There's a big calculus party, and all the functions are invited.

ln(x) is talking to some trig functions, when he sees his friend e^x sulking in a corner.

ln(x): "What's wrong e^x?"

e^x : "I'm so lonely!"

ln(x): "Well, you should go integrate yourself into the crowd!"

e^x looks up and cries, "It won't make a difference!"

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12.3 Rules of Integration

Integration of cosine function: $\int \cos \alpha x dx = \frac{\sin \alpha x}{\alpha} + C$

Integration of $\frac{1}{1+x^2}$ $\int \frac{dx}{1+x^2} = \arctan\ x + C$

<u>Note</u>: The rules of differentiation are complete, given a set of operations for constructing functions. But, the rules of integration are incomplete. We cannot integrate simple functions like sqrt($1+x^3$).

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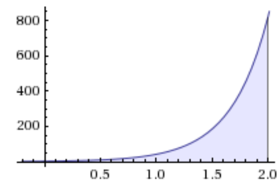
12.3 Rules of Integration: Example

- Evaluate the following: $\int_0^2 (x^3 + 2e^{3x} + 1)dx$
- Graph



- Sum of 3 integrals $\int_0^2 x^3 dx + 2\int_0^2 e^{3x} dx + \int_0^2 dx$
- Apply integration rules and FTC Part 1. $\left[\frac{1}{4}x^4\right]_0^2 + 2\left[\frac{e^{3x}}{3}\right]_0^2 + [x]_0^2$
- Compute area under the curve. $\frac{1}{4}\left(2^4\right) + \frac{2}{3}\left(e^6 - 1\right) + (2) = 274.29$

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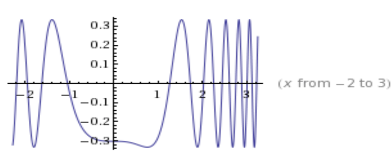
12.4 Integration by Substitution

- Some integrals cannot be easily solved by just applying the previous rules of integration.
- Consider $\int x^2 \cos(x^3 - 2)dx$
- Graph:



- We can use a chain rule like argument to simplify the integration. For example, let $u=x^3-2$, then $du=3x^2$ dx => x^2 dx = 1/3 du
- Substituting back into the original integral:

$$\int \frac{1}{3}\cos(u)du = \frac{1}{3}\sin(u) + C = \frac{1}{3}\sin(x^3 - 2) + C$$

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12.4 Integration by Substitution

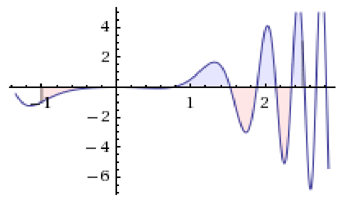
- Suppose we want to find the integral from -1 to 2.5.

  Then,

$$\int_{-1}^{2.5} x^2 \cos(x^3-2)dx = \left[= \frac{1}{3}\sin(x^3-2)+C\right]_{-1}^{2.5}$$

$$= \frac{1}{3}[\sin(2.5^3 - 2) - \sin(-1^3 - 2)] = 0.3376015$$

- Graph:



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12.4 Integration by Substitution: Rule

Theorem: Substitution Rule

Let *f(.)* be a continuous function defined on [*a, b*], and *s(.)* a continuously differentiable function from [*c, d*] into [*a, b*]. Then,

$$\int_a^b f(s(t))s'(t)\,dt = \int_{s(a)}^{s(b)} f(x)\,dx$$

• If we can identify a composition of functions as well as the derivative of one of the composed functions, we can find the antiderivative and evaluate the corresponding integral.

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12.4 Integration by Substitution - Example

- The key is to find the appropriate *u*. The variable of integration changes from *x* to *u*.

Note: It is important not to forget to substitute also d*x*.

- Another example:

$$\int (4x+3)^4\,dx$$
$$u = 4x+3$$
$$du = 4dx \Rightarrow \frac{du}{4} = dx$$
$$\int \frac{u^4}{4}\,du = \frac{1}{4}\frac{u^5}{5} + C$$

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12.5 Integration by Parts

- Recall the product rule of differentiation:
  d(*u v*) = *u dv* + *v du*
- Solve for *u* d*v*: $udv = d(uv) - vdu$
- Integrating both sides: $\int udv = \int [d(uv) - vdu]$
  $\int udv = uv - \int vdu$
- The last formula is used to *integrate by parts*.
- Key: Selection of *u* & *v* functions. In general, *u* involves logs, inverse, power, exponential, and trigonometric functions (in this order, LIPET).

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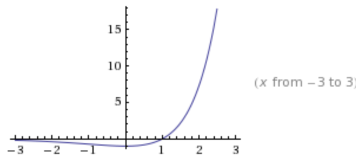
12.5 Integration by Parts: Example I

- Example: $\int xe^{x}\, dx$



- Select appropriate u & v functions –actually, select dv.
- Recall LIPET. We have a power function and an exponential function. Since power functions come before exponential functions, u equals x.
- From u, get du. => $u=x$, then $du=dx$
- From dv, get v. => $v=e^x$, then $dv=e^x dx$
- Then, simply plug this into the integration by parts formula.

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12.5 Integration by Parts: Example I (cont)

- Replace u, v, du, and dv.
- If the integral on the right looks easy to compute, then simply integrate it. Otherwise, you can integrate by parts again, or use substitution method.

$$\int udv = uv - \int vdu$$

$$\int xe^x dx = xe^x - \int e^x dx$$

$$= xe^x - e^x + C$$

$$= e^x(x-1) + C$$

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12.5 Integration by Parts: Example II

- Example: $\int e^{2x} \cos(x)dx$
- Select appropriate *u* & *v* functions.
- Exponential functions come before trigonometric functions. Then, $u = e^{2x}$.
- From *u*, get d*u*. => $u = e^{2x}$ =>then $du = 2e^{2x}dx$
- From *dv*, get *v*. => $dv = \cos(x)\,dx$ => $v = \sin(x)$.
- Then, simply plug this into the integration by parts formula.

$$K = \int udv = uv - \int vdu$$

$$\int e^{2x} \cos(x)dx = K$$

$$K = e^{2x}\sin(x) - 2\int e^{2x}\sin(x)dx$$

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12.5 Integration by Parts: Example II (cont)

- The expression looks more complicated than the original. Integrate by parts again.
- The integral of $e^{2x}\cos(x)$ is equal to K (the original integral).
- Replace the integral of $e^{2x}\cos(x)$ with K and solve for K.

$$K = e^{2x}\sin(x) - 2\int e^{2x}\sin(x)dx$$

$$u = e^{2x} \qquad \Rightarrow du = 2e^{2x}dx$$

$$v = -\cos(x) \qquad \Rightarrow dv = \sin(x)dx$$

$$K = e^{2x}\sin(x) - 2\left(-e^{2x}\cos(x) + 2\int e^{2x}\cos(x)dx\right)$$

$$\text{Note:} \quad K = \int e^{2x}\cos(x)dx$$

$$K = e^{2x}\sin(x) + 2e^{2x}\cos(x) - 4K$$

$$5K = e^{2x}\sin(x) + 2e^{2x}\cos(x)$$

$$K = \frac{e^{2x}\sin(x) + 2e^{2x}\cos(x)}{5} + C$$

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12.5 Integration by Parts: Tricks

- In an integral, do only integration by parts in the parts that are not easy to integrate.
- If the integration by parts is getting out of hand, you may have selected the wrong *u* function.
- If you see your original integral in the integral part of the integration by parts, just combine the two like integrals and solve for the integral.

- Integration by parts can be used to derive an effective way to compute the value of an integral numerically, the *trapezoid rule*.

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12.5 Integration by Parts: Trapezoid Rule

• Riemann sums can be used to approximate an integral, but convergence is slow. There are many rules designed to speed up the calculations. The trapezoid rule is simple, with good convergence.

• To prove it, we need the Mean Value Theorem for Integration

Theorem: MVT for Integration

Let f and g be continuous functions defined on $[a,b]$ so that $g(x) \geq 0$, then there exists a number $c \in [a,b]$ with

$$\int_a^b f(x)\, g(x)\, dx = f(c) \int_a^b g(x)\, dx$$

<u>Proof</u>: Simple exercise. (Use the supremum and infimum of $f(x)$ on $[a,b]$, apply Riemann integral properties and then use the Intermediate Value Theorem for continuous functions.)

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12.5 Integration by Parts: Trapezoid Rule

• Trapezoid Rule

Let $f(.)$ be a twice continuously differentiable function defined on $[a,b]$ and set

$K = sup\{\,|f''(x)|,\ x \in [a, b]\}.$

Define $h = (b - a)/n$, where n is a positive integer. Then,

$$\int_a^b f(x)dx = \left[\frac{1}{2} f(a) + \sum_{j=1}^{n-1} f(a + jh) + \frac{1}{2} f(b)\right] h + R(n)$$

$$= \frac{h}{2}\left[f(a) + 2\sum_{j=1}^{n-1} f(a + jh) + f(b)\right] h + R(n)$$

where $R(n) < \dfrac{K}{12}(b-a)h^2$

Note: The Trapezoid Rule is useful because the error, $R(n)$, depends on the square of h. If h is small, h^2 is a lot smaller!

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12.5 Integration by Parts: Trapezoid Rule

• Derivation of Trapezoid Rule

First, we prove the trapezoid rule on [0,1]. That is,

$$\int_0^1 f(x)dx = \left[\frac{1}{2} f(0) + \frac{1}{2} f(1)\right] - \frac{1}{12} f(c), \qquad \text{where } c \in [0,1].$$

Trick: Define a function

$v(x) = 1/2\, x\, (1 - x)$, which has the properties:

- $v(x) \geq 0$ for all $x \in [0,1]$ & $v(1) = v(0) = 0$.
- $v'(x) = 1/2 - x$ => $v'(1) = -1/2$ & $v(0) = ½$
- $v''(x) = -1$

Then,
$$\int_0^1 f(x)dx = -\int_0^1 v''(x) f(x)dx$$

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12.5 Integration by Parts: Trapezoid Rule

- $$\int_0^1 f(x)dx = -\int_0^1 v''(x) f(x)dx$$

We integrate by parts with $g'(x) = v''(x)$:

$\int v''(x) f(x)\, dx \quad = v'(1) f(1) - v'(0) f(0) - \int v'(x) f'(x)\, dx =$
$\qquad = -1/2\, f(1) - 1/2\, f(0) - \int v'(x) f'(x)\, dx$

Again, we integrate by parts with $g'(x) = v'(x)$:

$\int v'(x) f'(x)\, dx \quad = v(1) f'(1) - v(0) f'(0) - \int v(x) f''(x)\, dx$
$\qquad = - \int v(x) f''(x)\, dx = - f''(c) \int v(x)\, dx = - f''(c)\, 1/12$

where we used the MVT for Integration with some number $c \in [0,1]$.

Putting everything together, we have the simple trapezoid rule:

$\int f(x)\, dx \quad = 1/2\, f(1) + 1/2\, f(0) + \int v'(x) f'(x)\, dx =$
$\qquad = 1/2\, f(1) + 1/2\, f(0) - 1/12\, f''(c)$

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12.5 Integration by Parts: Trapezoid Rule

- General trapezoid rule: Assume that $f(.)$ is defined on $[a,b]$.

Let $h = (b - a)/n$, pick an integer j, and define the function

$u(x) = a + jh + xh \qquad$ for $x \in [0,1]$.

The composite function $g(x) = f(u(x))$ is twice continuously differentiable and defined on [0,1]. We can use the simple trapezoid rule:

$\int g(x)\, dx = 1/2\, g(0) + 1/2\, g(1) - 1/12\, g''(c)$

But $g(0)=f(u(0))=f(a+jh)$; $g(1)=f(u(1))=f(a + (j+1)h)$, & $g''(x)=h^2 f''(x)$.

Also notice that

$1/2\, g(0) + 1/2\, g(1) - 1/12\, g''(c) = \int g(x)\, dx = \int f(u(x))\, dx =$

$$= 1/h \int_0^1 f(u(x)) u'(x) dx = 1/h \int_{a+jh}^{a+(j+1)h} f(u) du$$

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12.5 Integration by Parts: Trapezoid Rule

• Then,

$$\int_{a+jh}^{a+(j+1)h} f(x)dx = \frac{h}{2}[f(a+jh)+f(a+(j+1)h)] - \frac{h^3}{12}f''(c)$$

• Summing this equation from j=0 to j=n-1 completes the derivation:

$$\int_a^b f(x)dx = h\left(\frac{1}{2}(f(a)+f(a+h)) + \frac{1}{2}(f(a+h)+f(a+2h)) + \dots\right.$$

$$\left.\dots + \frac{1}{2}(f(a+(n-2)h)+f(a+(n-1)h)) + \frac{1}{2}(f(a+(n-1)h)+f(a+nh))\right) + R(n) =$$

$$= h\left(\frac{1}{2}f(a)+f(a+h)+f(a+2h)+\dots+f(a+(n-1)h)+\frac{1}{2}f(b)\right) + R(n)$$

where

$$|R(n)| = \left|\sum_{j=0}^{n-1}\frac{h^3}{12}f''(c_j)\right| \le \frac{h^3}{12}\sum_{j=0}^{n-1}|f''(c_j)| \le \frac{K}{12}h^3 n = \frac{K}{12}(b-a)h^2$$

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12.6 Improper Riemann Integrals

• Improper Riemann integral: It is the limit of a definite integral as an endpoint of the interval(s) of integration approaches either a specified real number that causes a discontinuity or ∞ or −∞ or, in some cases, as both endpoints approach limits.

• Roughly, it is an integral that has infinity as its limits or has a discontinuity within its limits.

Examples:

$\int_1^{\infty} \frac{1}{x^2}dx$ Infinity as a boundary (Problem: domain of integration unbounded).

$\int_0^5 \frac{1}{x-4}dx$ Discontinuity at x=4 (Problem: integrand is unbounded in the domain of integration).

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12.6 Improper Riemann Integrals

• The Riemann integral can often be extended by continuity, by defining the improper integral instead as a limit.

• With limits of infinity, use a letter to replace the infinity, say τ, and treat it as a limit ($\lim_{\tau\to\infty}$). For example,

$$\int_2^{\infty} \frac{1}{x^2}\,dx = \lim_{\tau\to\infty} \int_2^{\tau} \frac{1}{x^2}\,dx = \lim_{\tau\to\infty}\left(-\frac{1}{\tau}+\frac{1}{2}\right) = \frac{1}{2}$$

• With points of discontinuity, split integral into parts. But, we cannot integrate to the point of discontinuity, say x_0. Then, we integrate to $x_0 \pm \delta$ and take limits as $\delta \rightarrow 0$.

• An improper integral converges if the limit defining it exists. It is also possible for an improper integral to diverge to infinity or to no particular value (oscillation).

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12.7 Applications: Value of a Bond

• Value of cash flows of a bond, paying a continuous dividend.
- Maturity: T (say, T=3 years)
- Face value: FV (say, \$1,000)
- Continuous discount rate: r (say, 5% annual)
- Continuous dividend rate: δ (say, 7% annual)

$$CF = \int_0^T \delta FV\, e^{-rt} dt = \delta\, FV \int_0^T e^{-rt} dt = \delta\, FV\left(-\frac{e^{-rt}}{r}\right)\Big|_0^T$$

$$= \frac{\delta}{r} FV(-e^{-rT} + e^{-0}) = \frac{\delta}{r} FV(1-e^{-rT})$$

$$CF(T=3; FV=1{,}000, r=.05, \delta=.07) = \frac{.07}{.05}\$1{,}000(1-e^{-.05x3}) = \$195.00$$

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12.7 Applications: Elements of Probability

- **X** is a random variable (an outcome of a random event). A random variable (RV) is a function.
- **X** can be discrete (recession, boom) or continuous (price of IBM stock tomorrow).
- Given a sample space **S** (**S:** set of all possible outcomes)
- To each discrete outcome **A** we associate a real number **P(A)**
- **P** is called a probability function and **P(A)** is called the probability of the event **A** if
  - Axiom 1: For every event **A**, $\mathbf{P(A) \geq 0}$
  - Axiom 2: For the Sure/Certain event **S**, $\mathbf{P(S) = 1}$
  - Axiom 3: For any number of mutually exclusive events $\mathbf{A_1}$, $\mathbf{A_2}$, $\mathbf{A_3}$ …, we have $\mathbf{P(A_1 \cup A_2 \cup A_3 \cup \ldots) = P(A_1) + P(A_2) + P(A_3) + \ldots}$

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12.7 Applications: Elements of Probability

- For continuous RV, the probability function is $f(x)$, such that:
  - $f(x) \geq 0$
  - $\int_{-\infty}^{\infty} f(x)dx = 1$
  - $P(a \leq X \leq b) = \int_a^b f(x)dx$
- $f(x)$ is called *probability density function* (pdf), or just *density*.
- The *cumulative distribution function* (CDF) of a discrete RV X, denoted as $F(x)$, is

  $F(x) = P(X \leq x) = \sum_{xi \leq x} P(x_i)$

  The cumulative distribution function (CDF) of a continuous RV X, denoted as F(x), is

  $F(x) = P(X \leq x) = \int_{xi \leq x} f(x_i)$

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12.7 Applications: Elements of Probability

Example: Uniform Distribution: $f(x)$=constant –i.e., the probability of any outcome is the same. Suppose x $\in$ [0,20]. What is the probability that x is between 10 and 15?

$$P(10 \leq X \leq 15) = \int_{10}^{15} f(x)dx = \int_{10}^{15} \frac{1}{20} dx = \frac{1}{20}\int_{10}^{15} dx = \frac{1}{20}(x)\Big|_{10}^{15} = \frac{1}{20}(15-10) = \frac{1}{4}.$$

Example: Exponential Distribution: $f(x)= \lambda e^{-\lambda x}$ for $0 \leq x \leq \infty$. Let Suppose λ =3, what is the probability that x is between 0 and 1?

$$P(0 \leq X \leq 1) = \int_{0}^{1} f(x)dx = 3\int_{0}^{1} e^{-3x}dx = -3\frac{e^{-3x}}{3} = -(e^{-3x})\Big|_{0}^{1}$$

$$= -(e^{-3} - 1) = 1 - e^{-3} = 0.9502$$

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12.7 Applications: Elements of Probability

- *Mean and Variance*

  Suppose X is a continuous RV with probability density function f(x). Then, the mean or expected value of X, denoted μ, is

$$\mu = E[X] = \int_{-\infty}^{\infty} x f(x)dx$$

- The variance of X, denoted as σ^2 or V[X], is

$$\sigma^2 = V[X] = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx$$

- The standard deviation, σ, is the square root of V[X].

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12.7 Applications: Elements of Probability

- Example: Uniform Distribution: $f(x) = 1/20$. Calculate the mean and variance of $f(x)$, where $x \in [0,20]$?

$$E[X] = \int_0^{20} x f(x) dx = \int_0^{20} \frac{x}{20} dx = \frac{1}{20}\int_0^{20} x dx = \frac{1}{20}(\frac{x^2}{2})\Big|_0^{20} = \frac{1}{40}(400) = 10.$$

$$V[X] = \int_0^{20} (x-\mu)^2 f(x) dx = \int_0^{20} \frac{(x-10)^2}{20} dx = \frac{1}{20}\int_0^{20} (u)^2 du = \frac{1}{20}(\frac{u^3}{3})\Big|_0^{20}$$

$$= \frac{1}{60}(x-10)^3\Big|_0^{20} = \frac{1}{60}(10^3 + 10^3) = 33.33$$

- Example: Exponential Distribution: $f(x) = \lambda\, e^{-\lambda x}$ for $0 \le x \le \infty$. Calculate the mean (integration by parts needed, $u=x$, $v=-e^{-\lambda x}$).

$$E[X] = \int_0^{\infty} x\lambda e^{-\lambda x} dx = e^{-\lambda x} x\Big|_0^{\infty} + \int_0^{\infty} e^{-\lambda x} dx = 0 - (\frac{e^{-\lambda x}}{\lambda})\Big|_0^{\infty} = \frac{1}{\lambda}.$$

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12.7 Applications: Truncated Normal

- Suppose we are interested in a regression,

$y_i = \mathbf{x}_i'\boldsymbol{\beta} + \varepsilon_i$, $\quad\varepsilon_i \sim N(0, \sigma^2)$

but we only observe the part of the sample with $y>0$.

Model: $y_i = \mathbf{x}_i'\boldsymbol{\beta} + \upsilon_i$, $\quad$ for $y_i = \mathbf{x}_i'\boldsymbol{\beta} + \upsilon_i > 0$

- Let's look at the density of υ_i, $f_\upsilon(.)$, which must integrate to 1:

$$\int_{-x_i'\beta}^{\infty} f_\upsilon(\eta)\, d\eta = 1$$

- The ε_i's density, normal by assumption:

$$\int_{-x_i'\beta}^{\infty} f_\varepsilon(\eta)\, d\eta = F_i = \int_{-\infty}^{x_i'\beta} f_\varepsilon(\eta)\, d\eta = \int_{-\infty}^{x_i'\beta} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}(\frac{\eta}{\sigma})^2}$$

- Then, $f_\upsilon(.)$ can be written as:

$$f_\upsilon = F_i^{-1} f_\varepsilon = F_i^{-1} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}(\frac{\upsilon_i}{\sigma})^2}$$

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12.7 Applications: Truncated Normal

• Now, we can calculate the expected value of υ_i:

$$E[\upsilon_i] = \int_{-x_i'\beta}^{\infty} \eta f_\upsilon(\eta) d\eta = F_i^{-1} \int_{-x_i'\beta}^{\infty} \eta f_\varepsilon(\eta) d\eta$$

$$= F_i^{-1} \int_{-x_i'\beta}^{\infty} \eta \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}(\frac{\eta}{\sigma})^2} d\eta = F_i^{-1} \int_{-x_i'\beta}^{\infty} \frac{\eta}{\sigma} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(\frac{\eta}{\sigma})^2} d\eta$$

$$= F_i^{-1} [-\sigma \ f_\varepsilon(\eta)]_{-x_i'\beta}^{\infty} \qquad \text{integration by substitution}$$

$$= \sigma \ F_i^{-1} f_i \neq 0 \qquad (f_i = f_\varepsilon(x_i'\beta))$$

- Integration by substitution: - $u = \eta/\sigma$ and $d\eta = \sigma \, du$.
- F(u) = exp($-u^2/2$)

- Then, $E[\upsilon_i|\mathbf{x}] = \sigma F_i^{-1} f_i = \sigma \lambda(\mathbf{x}_i'\boldsymbol{\beta}) \neq 0$ (and it depends on $\mathbf{x}_i'\beta$)
=> $E[y_i|y_i>0, \mathbf{x}_i'\boldsymbol{\beta}] = \mathbf{x}_i'\boldsymbol{\beta} + \sigma \lambda(\mathbf{x}_i'\boldsymbol{\beta})$
=> OLS in truncated part is *biased* (omitted variables problem).

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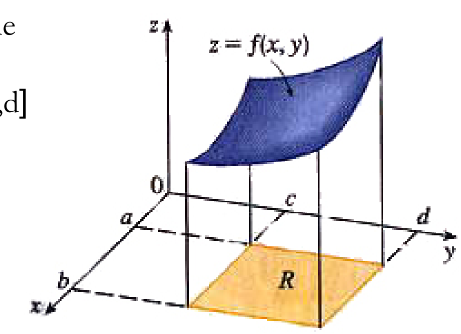
12.8 Double Integrals

- Now, $z=f(x,y)$, we have two variables of integration: x and y.
- Simple integrals represent areas, double integrals represent volume.

We want to know the volume defined by $z=f(x,y) \geq 0$ on the rectangle $R=[a,b]\times[c,d]$



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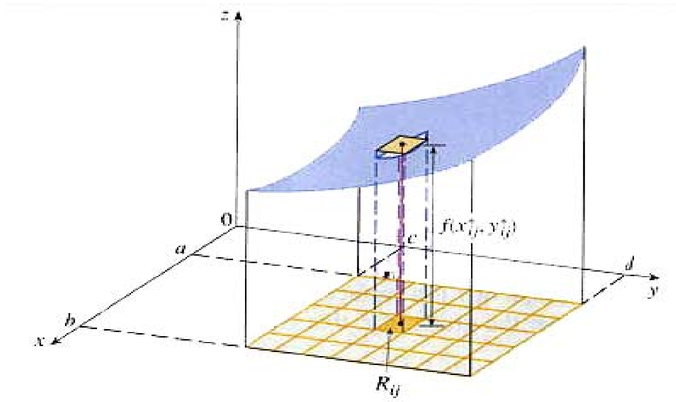
12.8 Double Integrals

• Similar to the intuition behind simple integrals, we can think of the double integral as a sum of small –easy to calculate- volumes.



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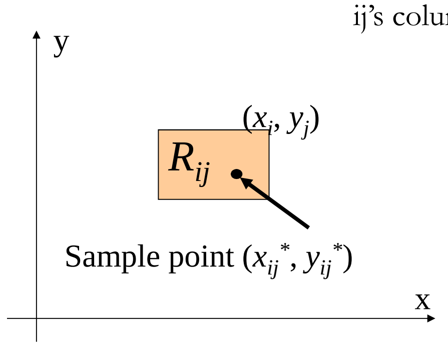
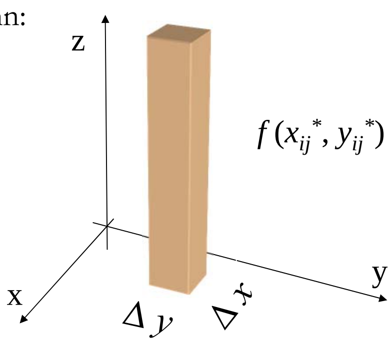
12.8 Double Integrals



ij's column:



Area of R_{ij} is $\Delta A = \Delta x \Delta y$

Volume of ij's column: $f(x_{ii}^*, y_{ii}^*)\Delta A$

Total volume of all columns: $\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A$

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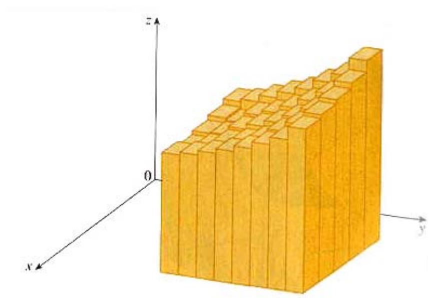
12.8 Double Integrals



$$V \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$

- Definition of a Double Integral:

$$V = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$

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12.8 Double Integrals

The double integral $\iint_R f(x,y)\,dA$ of f over the rectangle R is

$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A, \text{ if the limit exists.}$$

- Double Riemann sum: $\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$
- Note 1: If f is continuous then the limit exists and the integral is defined.
- Note 2: The definition of double integral does not depend on the choice of sample points.
- If the sample points are upper right-hand corners then

$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_i, y_j)\Delta A$$

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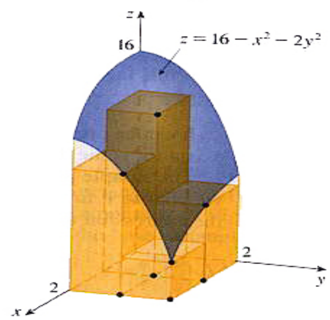
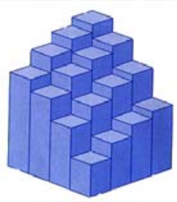
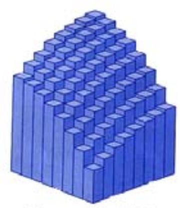
12.8 Double Integrals: Example

- Let z=16-x^2-2y^2, where 0≤x≤2 and 0≤y≤2.
- Estimate the volume of the solid above the square and below the graph
- Let's partitioned the volume in small (mxn) volumes.
- Exact volume: 48.





m=n=4;V≈41.5



m=n=8;V≈44.875

m=n=16;V≈ 46.46875

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12.8 Double Integrals: Fubini's Theorem

Theorem: Fubini's Theorem

Suppose A and B are complete measure spaces. Suppose $f(x,y)$ is $A \times B$ measurable. If

$$\int_{AxB} | f(x,y) | \, d(x,y) < \infty$$

where the integral is taken with respect to a product measure on the space over $A \times B$, then

$$\iint_{AxB} f(x,y)dA = \int_A \left(\int_B f(x,y)dx \right) dy = \int_B \left(\int_A f(x,y)dy \right) dx$$

where the last two integrals are iterated integrals with respect to two measures, respectively, and the first being an integral with respect to a product of these two measures.

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12.8 Double Integrals: Fubini's Theorem

Fubini's Theorem is very general. For the Riemann's case, we have:

If $f(x,y)$ is continuous on rectangle R=[a,b]×[c,d] then the double integral is equal to the *iterated integral*.

$$\iint_R f(x,y)dA = \int_c^d \int_a^b f(x,y)dxdy = \int_a^b \int_c^d f(x,y)dydx$$

That is, we can compute first

$$\int_a^b f(x,y)dx$$

by holding y constant and integrating over x as if this were an single integral. This creates a function with only x, which we can integrate as usual. Then, we integrate over y, again, as usual.

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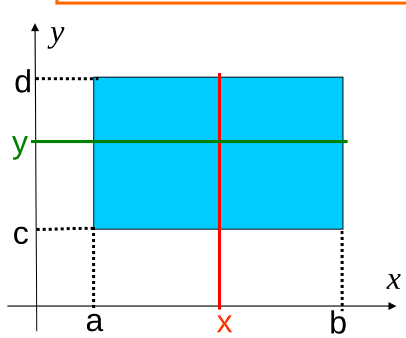
12.8 Double Integrals: Computation

• Fubini's theorem simplifies calculation by allowing iterated computations. There are two ways of doing the iteration:

$$\boxed{\iint_R f(x,y)dA = \int_c^d \int_a^b f(x,y)dxdy = \int_a^b \int_c^d f(x,y)dydx}$$

fixed

fixed



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12.8 Double Integrals: Computation

- Example:

$$\iint_R f(x,y)dA = \int_0^1\int_0^1 (x+y)dxdy = \int_0^1\left(\int_0^1 (x+y)dx\right)dy =$$

$$= \int_0^1 \left(\frac{x^2}{2} + xy\right)_0^1 dy = \int_0^1 \left(\frac{1}{2} + y\right)dy = \int_0^1 \frac{1}{2}dy + \int_0^1 ydy$$

$$= \left(\frac{y}{2}\right)_0^1 + \left(\frac{y^2}{2}\right)_0^1 = \frac{1+1}{2} = 1$$

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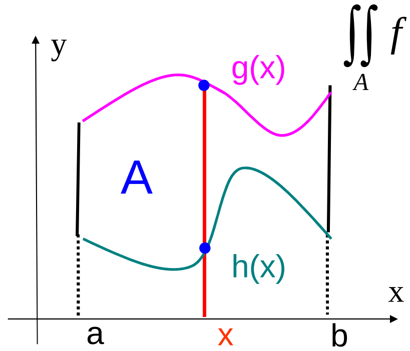
12.8 Double Integrals: Computation (General Case)

- Before, we looked at double integrals over a rectangular region, *R*. Not realistic. Most regions are not rectangular. We adapt our previous result to the general case.
- If $f(x,y)$ is continuous on $A = \{(x,y) \mid x \in [a,b] \ \& \ h(x) \le y \le g(x)\}$, then the double integral is equal to the iterated integral:

$$\iint_A f(x,y)dA = \int_a^b \int_{h(x)}^{g(x)} f(x,y)dydx$$



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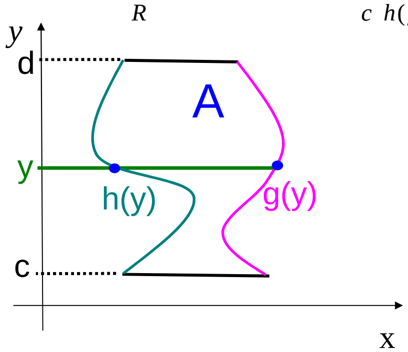
12.8 Double Integrals: Computation (General Case)

- Similarly, if $f(x,y)$ is continuous on $A = \{(x,y) \mid y \in [c,d] \;\&\; h(y) \le x \le g(y)\}$ then the double integral is equal to the iterated integral:

$$\iint_R f(x,y)\,dA = \int_c^d \int_{h(y)}^{g(y)} f(x,y)\,dx\,dy$$



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12.8 Double Integrals: Fubini's Thorem Corollary

- If $f(x,y) = \varphi(x)\,\psi(y)$ then

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b \varphi(x)\psi(y)\,dx\,dy = \left[\int_a^b \varphi(x)\,dx\right] \cdot \left[\int_c^d \psi(y)\,dy\right]$$

Examples:

$$\iint_R y\sin(x)\,dA, \quad A = [1/2,1] \times [\pi/2, \pi]$$

$$\iint_R \frac{1}{2\pi} e^{-\frac{(x-\mu_x)^2}{2}} e^{-\frac{(y-\mu_y)^2}{2}}\,dx\,dy, \quad R = [-\infty,\infty] \times [-\infty,\infty]$$

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12.8 Double Integrals: Polar Coordinates

• Sometimes, it is easier to move from Cartesian coordinates to polar coordinates. For example, we have a region that is a disk or a portion of a ring. Cartesian coordinates could be cumbersome.

Examples: $\iint_D f(x,y)dA$ where D is a disk of radius 2.

We can describe the area D as: $-2 \leq x \leq 2$ & $-\sqrt{(4-x^2)} \leq y \leq \sqrt{(4-x^2)}$

$$\iint_D f(x,y)dA = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{-\sqrt{4-x^2}} f(x,y)dxdy$$

Easier to describe a disk of radius 2 in polar coordinates:

$$0 \leq \theta \leq 2\pi \quad \& \quad 0 \leq r \leq 2$$

To integrate, we need a change of variables: $x = r\sin(\theta)$, $y = r\cos(\theta)$ and $dA = r\,dr\,d\theta$.

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12.8 Double Integrals: Polar Coordinates

• Let's generalize the example. Now,:

$$\alpha \leq \theta \leq \beta \qquad \& \qquad h_1(\theta) \leq r \leq h_2(\theta)$$

Then

$$\iint_D f(x,y)dA = \int_{\alpha}^{\beta} \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos(\theta), r\sin(\theta))rdrd\theta$$

Note: $dA = r\,dr\,d\theta$ (not $dA = dr\,d\theta$, as in the Cartesian world).

Example:

$$\iint_{D=R^2} e^{-(\frac{x^2}{2}+\frac{y^2}{2})}dxdy = \int_0^{2\pi} \int_0^{\infty} e^{-(\frac{r^2}{2})} rdrd\theta = \int_0^{2\pi} \left[-e^{-(\frac{r^2}{2})}\right]_0^{\infty} d\theta = \int_0^{2\pi} 1d\theta = 2\pi$$

We use a change of variables: $x = r\sin(\theta)$, $y = r\cos(\theta)$ (recall $r^2 = x^2 + y^2$) and $dA = r\,dr\,d\theta$.

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12.8 Double Integrals: Properties

• Linearity

$$\iint_A [f(x,y)+g(x,y)]dA = \iint_A f(x,y)dA + \iint_A g(x,y)dA$$

$$\iint_A cf(x,y)dA = c\iint_A f(x,y)dA$$

• Comparison: If $f(x,y) \geq g(x,y)$ for all (x,y) in R, then

$$\iint_A f(x,y)dA \geq \iint_A g(x,y)dA$$

• Additivity: If A_1 and A_2 are non-overlapping regions then

$$\iint_{A_1 \cup A_2} f(x,y)dA = \iint_{A_1} f(x,y)dA + \iint_{A_2} f(x,y)dA$$

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12.9 Computational Science vs. Calculus

- Calculus tells you how to compute precise integrals & derivatives when you know the equation (analytical form) for a problem; for example, for the indefinite integral:

  $\int(-t^2 + 10t + 24)\ dt = -t^3/3 + 5t^2 + 24t + C$

- It turns out that many integral do not have analytical solutions or are complicated to compute, especially when we move to more than 3 dimensions. For these problems, we rely on numerical approximations.

- Computational science provides methods for *estimating* integrals and derivatives from actual data.

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