

Evaluating Definite Integrals

THEOREM (Evaluation Theorem): Suppose f is integrable on $[a, b]$ and $f = F'$ for an other function F . Then

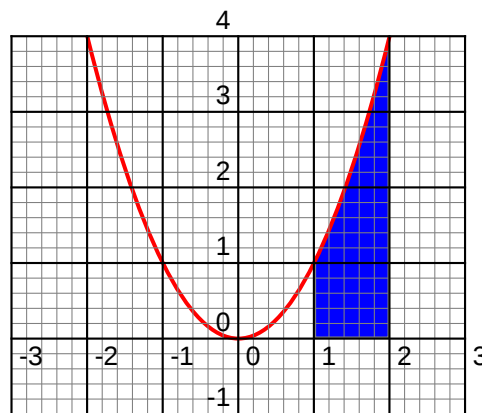
$$\int_a^b f(x)dx = F(b) - F(a) = \left. F(x) \right|_a^b$$

EXAMPLES:

$$1. \int_1^2 x^2 dx = \left. \frac{x^3}{3} \right|_1^2 = \frac{2^3}{3} - \frac{1^3}{3} = \frac{7}{3}$$

$$2. \int_0^{\pi/2} \sin x dx = \left. -\cos x \right|_0^{\pi/2} = -\cos(\pi/2) - (-\cos(0)) \\ = 0 - (-1) = 1$$

$$3. \int_2^3 \frac{1}{x} dx = \left. \ln |x| \right|_2^3 = \ln 3 - \ln 2 = 0.4054\dots$$



THEOREM: If F is an antiderivative of f on an interval I , then the most general antiderivative of f on I is $F(x) + C$, where C is an arbitrary constant.

NOTATION: To denote the set of all antiderivatives of f we use the indefinite integral notation:

$$\int f(x)dx = F(x) + C$$

Table of Indefinite Integrals

$\int cf(x)dx = c \int f(x)dx$	$\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$
$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$	$\int \frac{1}{x} dx = \ln x + C$
$\int e^x dx = e^x + C$	$\int a^x dx = \frac{a^x}{\ln a} + C$
$\int \sin x dx = -\cos x + C$	$\int \cos x dx = \sin x + C$
$\int \sec^2 x dx = \tan x + C$	$\int \csc^2 x dx = -\cot x + C$
$\int \sec x \tan x dx = \sec x + C$	$\int \csc x \cot x dx = -\csc x + C$
$\int \frac{1}{1+x^2} dx = \arctan x + C$	$\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + C$

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EXAMPLES:

- $\int x^2 dx = [PR \text{ with } n = 2] = \frac{x^{2+1}}{2+1} + C = \frac{x^3}{3} + C$
- $\int x^{-8} dx = [PR \text{ with } n = -8] = \frac{x^{-8+1}}{-8+1} + C = \frac{x^{-7}}{-7} + C = -\frac{1}{7}x^{-7} + C$
- $\int \sqrt{x} dx = \int x^{1/2} dx = [PR \text{ with } n = 1/2] = \frac{x^{1/2+1}}{1/2+1} + C = \frac{x^{3/2}}{3/2} + C = \frac{2}{3}x^{3/2} + C$
- $\int \frac{1}{\sqrt[5]{x}} dx = \int \frac{1}{x^{1/5}} dx = \int x^{-1/5} dx = [PR \text{ with } n = -1/5] = \frac{x^{-1/5+1}}{-1/5+1} + C = \frac{5}{4}x^{4/5} + C$
- $\int x\sqrt[3]{x} dx = \int x \cdot x^{1/3} dx = \int \underbrace{x^{1+1/3}}_{x^{4/3}} dx = [PR \text{ with } n = 4/3] = \frac{x^{4/3+1}}{4/3+1} + C = \frac{3}{7}x^{7/3} + C$
- $\int \frac{x^4}{\sqrt[5]{x^3}} dx = \int \frac{x^4}{x^{3/5}} dx = \int \underbrace{x^{4-3/5}}_{x^{17/5}} dx = [PR \text{ with } n = 17/5] = \frac{x^{17/5+1}}{17/5+1} + C = \frac{5}{22}x^{22/5} + C$
- $\int \frac{x\sqrt[3]{x}}{\sqrt{x}} dx$

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$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$	$\int \frac{1}{x} dx = \ln x + C$
$\int e^x dx = e^x + C$	$\int a^x dx = \frac{a^x}{\ln a} + C$
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$$\begin{aligned}
 7. \quad \int \frac{x\sqrt[3]{x}}{\sqrt{x}} dx &= \int \frac{x \cdot x^{1/3}}{x^{1/2}} dx = \int \frac{x^{1+1/3}}{x^{1/2}} dx = \int \underbrace{x^{1+1/3-1/2}}_{x^{5/6}} dx = [PR \text{ with } n = 5/6] = \frac{x^{5/6+1}}{5/6+1} + C \\
 &= \frac{6}{11} x^{11/6} + C
 \end{aligned}$$

$$8. \quad \int x(1+x^2) dx = \int (x+x^3) dx = \int x dx + \int x^3 dx = \frac{x^2}{2} + \frac{x^4}{4} + C$$

$$\begin{aligned}
 9. \quad \int_0^1 \sqrt{x} \left(\sqrt[3]{x\sqrt{x}} + \sqrt[4]{x} \right) dx &= \int_0^1 x^{1/2} \left((x \cdot x^{1/2})^{1/3} + x^{1/4} \right) dx = \int_0^1 x^{1/2} \left((x^{1+1/2})^{1/3} + x^{1/4} \right) dx \\
 &= \int_0^1 x^{1/2} \left((x^{3/2})^{1/3} + x^{1/4} \right) dx = \int_0^1 x^{1/2} (x^{3/2 \cdot 1/3} + x^{1/4}) dx = \int_0^1 x^{1/2} (x^{1/2} + x^{1/4}) dx \\
 &= \int_0^1 (x^{1/2} \cdot x^{1/2} + x^{1/2} \cdot x^{1/4}) dx = \int_0^1 (x^{1/2+1/2} + x^{1/2+1/4}) dx = \int_0^1 (x + x^{3/4}) dx \\
 &= \left[\frac{x^2}{2} + \frac{x^{3/4+1}}{3/4+1} \right]_0^1 = \left[\frac{x^2}{2} + \frac{x^{7/4}}{7/4} \right]_0^1 = \left[\frac{1}{2} x^2 + \frac{4}{7} x^{7/4} \right]_0^1 \\
 &= \left(\frac{1}{2} \cdot 1^2 + \frac{4}{7} \cdot 1^{7/4} \right) - \left(\frac{1}{2} \cdot 0^2 + \frac{4}{7} \cdot 0^{7/4} \right) = \frac{1}{2} + \frac{4}{7} = \frac{15}{14}
 \end{aligned}$$

$$10. \quad \int x(1+x^2)^2 dx$$

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$\int cf(x)dx = c \int f(x)dx$	$\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$
$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$	$\int \frac{1}{x} dx = \ln x + C$
$\int e^x dx = e^x + C$	$\int a^x dx = \frac{a^x}{\ln a} + C$
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$\int \frac{1}{1+x^2} dx = \arctan x + C$	$\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + C$

$$\begin{aligned}
 10. \quad \int x(1+x^2)^2 dx &= \int x(1+2x^2+x^4) dx = \int (x+2x^3+x^5) dx \\
 &= \int x dx + 2 \int x^3 dx + \int x^5 dx = \frac{x^2}{2} + 2 \frac{x^4}{4} + \frac{x^6}{6} + C = \frac{1}{2}x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6 + C
 \end{aligned}$$

$$\begin{aligned}
 11. \quad &\int \left(\frac{11x^2 - 5x^{-1/2} + 4x - 2}{7\sqrt{x}} \right) dx = \int \left(\frac{11x^2 - 5x^{-1/2} + 4x - 2}{7x^{1/2}} \right) dx \\
 &= \int \left(\frac{11x^2}{7x^{1/2}} - \frac{5x^{-1/2}}{7x^{1/2}} + \frac{4x}{7x^{1/2}} - \frac{2}{7x^{1/2}} \right) dx = \int \left(\frac{11}{7}x^{2-1/2} - \frac{5}{7}x^{-1/2-1/2} + \frac{4}{7}x^{1-1/2} - \frac{2}{7}x^{-1/2} \right) dx \\
 &= \int \left(\frac{11}{7}x^{3/2} - \frac{5}{7}x^{-1} + \frac{4}{7}x^{1/2} - \frac{2}{7}x^{-1/2} \right) dx = \frac{11}{7} \int x^{3/2} dx - \frac{5}{7} \int x^{-1} dx + \frac{4}{7} \int x^{1/2} dx - \frac{2}{7} \int x^{-1/2} dx \\
 &= \frac{11}{7} \cdot \frac{x^{3/2+1}}{3/2+1} - \frac{5}{7} \ln|x| + \frac{4}{7} \cdot \frac{x^{1/2+1}}{1/2+1} - \frac{2}{7} \cdot \frac{x^{-1/2+1}}{-1/2+1} + C \\
 &= \frac{11}{7} \cdot \frac{x^{5/2}}{5/2} - \frac{5}{7} \ln|x| + \frac{4}{7} \cdot \frac{x^{3/2}}{3/2} - \frac{2}{7} \cdot \frac{x^{1/2}}{1/2} + C \\
 &= \frac{11}{7} \cdot \frac{2}{5}x^{5/2} - \frac{5}{7} \ln|x| + \frac{4}{7} \cdot \frac{2}{3}x^{3/2} - \frac{2}{7} \cdot 2x^{1/2} + C \\
 &= \frac{22}{35}x^{5/2} - \frac{5}{7} \ln|x| + \frac{8}{21}x^{3/2} - \frac{4}{7}x^{1/2} + C
 \end{aligned}$$

$$12. \quad \int \left(\frac{4 - 5\sqrt{x} + 7x \sin x}{9x} \right) dx$$

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$$\begin{aligned}
12. \quad & \int \left(\frac{4 - 5\sqrt{x} + 7x \sin x}{9x} \right) dx = \int \left(\frac{4}{9x} - \frac{5\sqrt{x}}{9x} + \frac{7x \sin x}{9x} \right) dx = \int \left(\frac{4}{9} \cdot \frac{1}{x} - \frac{5}{9} x^{-1/2} + \frac{7}{9} \sin x \right) dx \\
& = \int \frac{4}{9} \cdot \frac{1}{x} dx - \int \frac{5}{9} x^{-1/2} dx + \int \frac{7}{9} \sin x dx = \frac{4}{9} \int \frac{1}{x} dx - \frac{5}{9} \int x^{-1/2} dx + \frac{7}{9} \int \sin x dx \\
& = \frac{4}{9} \ln |x| - \frac{5}{9} \cdot \frac{x^{-1/2+1}}{-1/2+1} - \frac{7}{9} \cos x + C = \frac{4}{9} \ln |x| - \frac{10}{9} x^{1/2} - \frac{7}{9} \cos x + C
\end{aligned}$$

$$13. \quad \int_0^1 \frac{x^3 + x + 1}{x^2 + 1} dx$$

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$$\begin{aligned}
 13. \quad \int_0^1 \frac{x^3 + x + 1}{x^2 + 1} dx &= \int_0^1 \frac{x(x^2 + 1) + 1}{x^2 + 1} dx = \int_0^1 \left(\frac{x(x^2 + 1)}{x^2 + 1} + \frac{1}{x^2 + 1} \right) dx = \int_0^1 \left(x + \frac{1}{x^2 + 1} \right) dx \\
 &= \left[\frac{x^2}{2} + \arctan x \right]_0^1 = \left(\frac{1^2}{2} + \arctan 1 \right) - \left(\frac{0^2}{2} + \arctan 0 \right) = \frac{1}{2} + \arctan 1 = \frac{1}{2} + \frac{\pi}{4}
 \end{aligned}$$

$$\begin{aligned}
 14. \quad \int 2 \sec x (3 \sec x - 5 \tan x) dx &= \int (6 \sec^2 x - 10 \sec x \tan x) dx = 6 \int \sec^2 x dx - 10 \int \sec x \tan x dx \\
 &= 6 \tan x - 10 \sec x + C
 \end{aligned}$$

$$15. \quad \int \frac{\sin 2x}{\sin x} dx$$

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$$15. \int \frac{\sin 2x}{\sin x} dx = \int \frac{2 \sin x \cos x}{\sin x} dx = \int 2 \cos x dx = 2 \int \cos x dx = 2 \sin x + C$$

$$16_1. \int_0^{\pi/3} \frac{\cos x + \cos x \tan^2 x}{2 \sec^2 x} dx = \int_0^{\pi/3} \frac{\cos x(1 + \tan^2 x)}{2 \sec^2 x} dx = \int_0^{\pi/3} \frac{\cos x \sec^2 x}{2 \sec^2 x} dx = \int_0^{\pi/3} \frac{\cos x}{2} dx$$

$$= \frac{1}{2} \int_0^{\pi/3} \cos x dx = \frac{1}{2} \sin x \Big|_0^{\pi/3} = \frac{1}{2} \left(\sin \left(\frac{\pi}{3} \right) - \sin 0 \right) = \frac{1}{2} \left(\frac{\sqrt{3}}{2} - 0 \right) = \frac{\sqrt{3}}{4}$$

$$16_2. \int_0^{\pi/3} \frac{\cos x + \cos x \tan^2 x}{2 \sec^2 x} dx = \int_0^{\pi/3} \frac{\cos x(1 + \tan^2 x)}{2 \sec^2 x} dx = \int_0^{\pi/3} \frac{\cos x \left(1 + \frac{\sin^2 x}{\cos^2 x} \right)}{2 \cdot \frac{1}{\cos^2 x}} dx$$

$$= \int_0^{\pi/3} \frac{\cos x \left(\frac{\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} \right)}{2 \cdot \frac{1}{\cos^2 x}} dx = \int_0^{\pi/3} \frac{\cos x \left(\frac{\cos^2 x + \sin^2 x}{\cos^2 x} \right)}{2 \cdot \frac{1}{\cos^2 x}} dx = \int_0^{\pi/3} \frac{\cos x \cdot \frac{1}{\cos^2 x}}{2 \cdot \frac{1}{\cos^2 x}} dx = \int_0^{\pi/3} \frac{\cos x}{2} dx$$

$$= \frac{1}{2} \int_0^{\pi/3} \cos x dx = \frac{1}{2} \sin x \Big|_0^{\pi/3} = \frac{1}{2} \left(\sin \left(\frac{\pi}{3} \right) - \sin 0 \right) = \frac{1}{2} \left(\frac{\sqrt{3}}{2} - 0 \right) = \frac{\sqrt{3}}{4}$$

$$17. \int (x^2 + 2^x) dx = \int x^2 dx + \int 2^x dx = \frac{x^3}{3} + \frac{2^x}{\ln 2} + C$$

Applications

THEOREM (Net Change Theorem): The integral of a rate of change is the net change:

$$\int_a^b F'(x)dx = F(b) - F(a)$$

EXAMPLE: We have

$$\int_a^b x^2 dx = \frac{b^3}{3} - \frac{a^3}{3} \implies \int_a^b \left(\frac{x^3}{3}\right)' dx = \frac{b^3}{3} - \frac{a^3}{3}$$

APPLICATIONS:

(a) If an object moves along a straight line with position function $s(t)$, then its velocity is $v(t) = s'(t)$, so

$$\int_{t_1}^{t_2} v(t)dt = s(t_2) - s(t_1)$$

is the net change of position, or *displacement*, of the particle during the time period from t_1 to t_2 . Similarly,

$$\int_{t_1}^{t_2} |v(t)|dt = \text{total distance traveled}$$

(b) The acceleration of the object is $a(t) = v'(t)$, so

$$\int_{t_1}^{t_2} a(t)dt = v(t_2) - v(t_1)$$

is the change in velocity from time t_1 to time t_2 .

EXAMPLE: A particle moves along a line so that its velocity at time t is $v(t) = t^2 - t - 6$ (measured in meters per second).

(a) Find the displacement of the particle during the time period $1 \leq t \leq 4$.

(b) Find the distance traveled during this time period.

EXAMPLE: A particle moves along a line so that its velocity at time t is $v(t) = t^2 - t - 6$ (measured in meters per second).

(a) Find the displacement of the particle during the time period $1 \leq t \leq 4$.

(b) Find the distance traveled during this time period.

Solution:

(a) The displacement is

$$s(4) - s(1) = \int_1^4 v(t) dt = \int_1^4 (t^2 - t - 6) dt = \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t \right]_1^4 = -\frac{9}{2}$$

This means that the particle's position at time $t = 4$ is 4.5 m to the left of its position at the start of the time period.

(b) Note that

$$v(t) = t^2 - t - 6 = (t - 3)(t + 2)$$

and so $v(t) \leq 0$ on the interval $[1, 3]$ and $v(t) \geq 0$ on $[3, 4]$. Thus the distance traveled is

$$\begin{aligned} \int_1^4 |v(t)| dt &= \int_1^3 [-v(t)] dt + \int_3^4 v(t) dt = \int_1^3 (-t^2 + t + 6) dt + \int_3^4 (t^2 - t - 6) dt \\ &= \left[-\frac{t^3}{3} + \frac{t^2}{2} + 6t \right]_1^3 + \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t \right]_3^4 = \frac{61}{6} \approx 10.17 \text{ m} \end{aligned}$$