Lebesgue Measure
on the real line

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The **length** of a bounded interval I (open, closed, half-open) with endpoints a and b ($a < b$) is defined by $\ell(I) := b - a$. If I is (a, ∞), $(-\infty, b)$, or $(-\infty, \infty)$, then $\ell(I) = \infty$. Is it possible to extend this concept of length (or *measure*) to arbitrary subsets of $\mathbb{R}$? When one attempts to do this, one is led rather naturally to what has become known as **Lebesgue measure**, named after Henri Lebesgue [1875–1941] who was one of the first people to carry this out in complete detail and to apply it with great success to many problems of real and complex analysis, especially to the theories of Fourier series & integrals. It led von Neumann [1903–1957] and his followers, most notably Alain Connes, to the astounding and vastly more general theory called **noncommutative geometry** where measure theory evolved via the **spectral theory of operators on Hilbert space** to **von Neumann algebras** with applications to diverse parts of mathematics & physics.

Given a set E of real numbers, $\mu(E)$ will denote its **Lebesgue measure** if it's defined. Here are the properties we wish it to have.

(1) **Extends length**: For every interval I, $\mu(I) = \ell(I)$.

(2) **Monotone**: If $A \subset B \subset \mathbb{R}$, then $0 \le \mu(A) \le \mu(B) \le \infty$.

(3) **Translation invariant**: For each subset A of $\mathbb{R}$ and for each point $x_0 \in \mathbb{R}$ we define $A + x_0 := \{x + x_0 : x \in A\}$. Then $\mu(A + x_0) = \mu(A)$.

(4) **Countably additive**: If A and B are *disjoint* subsets of $\mathbb{R}$, then $\mu(A \cup B) = \mu(A) + \mu(B)$. If $\{A_i\}$ is a sequence of disjoint sets, then $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i)$.

It turns out that it is not possible to define a function $\mu : 2^{\mathbb{R}} \to [0, \infty]$ that satisfies all of these properties. It is possible if we allow our measure to assume "infinitesimal" values; that is, if we take $\mu : 2^{\mathbb{R}} \to [0, \infty]^*$, where $[0, \infty]^* \subset \mathbb{R}^*$, a **nonstandard model** of $\mathbb{R}$. But this possibility is not discussed any further here. Rather, we determine the largest family $\mathcal{M}(\mathbb{R})$ of subsets of $\mathbb{R}$ for which (1)–(4) can hold with $\mu : \mathcal{M} \to [0, \infty]$. Members of $\mathcal{M} = \mathcal{M}(\mathbb{R})$ are called the **Lebesgue measurable** subsets of $\mathbb{R}$. Subsets of $\mathbb{R}$ that are not members of $\mathcal{M}$ are called **nonmeasurable** sets. We proceed as follows to define μ.

Definition 1 *For each subset E of $\mathbb{R}$ we define its* **Lebesgue outer measure** $\mu^*(E)$ *by*

$$\mu^*(E) := \inf\{\sum_{k=1}^{\infty} \ell(I_k) : \{I_k\} \text{ a sequence of open intervals with } E \subset \bigcup_{k=1}^{\infty} I_k\}.$$

It is obvious that $0 \le \mu^*(E) \le \infty$ for every set $E \subset \mathbb{R}$.

Theorem 1 *Lebesgue outer measure $\mu^*(E)$ is zero if E is countable; extends length; is monotone & translation invariant; and is* **countably subadditive:** $\forall$ *sequence $E_i \subset \mathbb{R}$,*

$$\mu^*\left(\bigcup_{i=1}^{\infty} E_i\right) \le \sum_{i=1}^{\infty} \mu^*(E_i).$$

That is, every subset of $\mathbb{R}$ has Lebesgue *outer* measure which satisfies properties (1)–(3), but satisfies only part of property (4). Examples of disjoint sets A and B for which $\mu^*(A \cup B) \neq \mu^*(A) + \mu^*(B)$ seem at first a bit bizarre. Such an example is given below. But first we proceed to define the class $\mathcal{M}$ of measurable sets—those for which (4) holds as well as (1)–(3).

Definition 2 *A set $E \subset \mathbb{R}$ is called* **Lebesgue measurable** *if for every subset A of $\mathbb{R}$,*

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap \complement E).$$

Definition 3 *If E is a Lebesgue measurable set, then the* **Lebesgue measure** *of E is defined to be its outer measure $\mu^*(E)$ and is written $\mu(E)$.*

Theorem 2 *The collection $\mathcal{M}$ of Lebesgue measurable sets has the following properties:*

(a) *Both $\emptyset$ and $\mathbb{R}$ are measurable; $\mu(\emptyset) = 0$ and $\mu(\mathbb{R}) = \infty$.*

(b) *If E is measurable, then so is $\complement E$.*

(c) *If $\mu^*(E) = 0$, then E is measurable.*

(d) *If E_1 and E_2 are measurable, then $E_1 \cup E_2$ and $E_1 \cap E_2$ are measurable.*

(e) *If E is measurable, then $E + x_0$ is measurable.*

(f) *Every interval is measurable and $\mu(I) = \mu^*(I) = \ell(I)$.*

(g) *If $\{ E_i : 1 \leq i \leq n \}$ is a finite collection of disjoint measurable sets, then $\forall A \subset \mathbb{R}$,*

$$\mu^*\left(\bigcup_{i=1}^{n} A \cap E_i\right) = \mu^*\left(A \cap \left(\bigcup_{i=1}^{n} E_i\right)\right) = \sum_{i=1}^{n} \mu^*(A \cap E_i).$$

In particular, when $A = \mathbb{R}$, we have

$$\mu\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} \mu(E_i).$$

(h) *If $\{E_i\}$ is a sequence of measurable sets, then*

$$\bigcup_{i=1}^{\infty} E_i \text{ and } \bigcap_{i=1}^{\infty} E_i$$

are also measurable sets.

(i) *If $\{E_i\}$ is an arbitrary sequence of disjoint measurable sets, then*

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i).$$

(j) *Every open set and every closed set is measurable.*

Remark 1 *Lebesgue measure $\mu(E)$ satisfies the properties (1)–(4) on the collection $\mathcal{M}$ of measurable subsets of $\mathbb{R}$. However, not all subsets of $\mathbb{R}$ are measurable.*

Theorem 3 *Properties (1)–(4) imply there exist* **nonmeasurable sets**. $\mathcal{M} \neq 2^{\mathbb{R}}$.

Since all open sets and all closed sets are measurable, and the family $\mathcal{M}$ of measurable sets is closed under countable unions and countable intersections, it is hard to imagine a set that is not measurable. However many such sets do exist!

Proof of Theorem 3 (An example of a nonmeasurable subset of $\mathbb{R}$):
Define the relation $x \sim y$ on $\mathbb{R}$ to mean that $x - y \in \mathbb{Q}$. One easily checks that $x \sim y$ is an **equivalence relation** on $\mathbb{R}$; reflexive, symmetric, and transitive. Therefore this relation $x \sim y$ establishes a collection of **equivalence classes** of the form $\{ x + r : r \in \mathbb{Q} \}$. Each equivalence class contains a point in the interval $[0, 1]$. Let $E \subset [0, 1]$ be a set that consists of exactly one point from each equivalence class. (Note the use here of the **Axiom of Choice**.) Let $\{ r_i : i = 1, 2, 3, \ldots \}$ be an enumeration of the denumerable set of all rational numbers in the closed interval $[-1, 1]$, and let $E_i = E + r_i$ for each $i = 1, 2, 3, \ldots$. We claim that

$$[0,1] \subset \bigcup_{i=1}^{\infty} E_i \subset [-1,2].$$

The second inclusion is obvious. To prove the first, let $x \in [0, 1]$. Then there exists $y \in E$ such that $x - y$ is rational. So $x = y + r_j \in E_j$. Furthermore, $E_i \cap E_j = \emptyset$ whenever $i \neq j$. For otherwise, there would exist y and z in E such that $y + r_i = z + r_j$ which would imply that $y \sim z$, a contradiction. Now suppose that E is a measurable set. Then each E_i is measurable and $\mu(E_i) = \mu(E)$ by translation-invariance (3). But Theorem 2(i) yields

$$1 \leq \mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E) \leq 3,$$

which is impossible. Therefore the set E is not measurable. $\square$

The above proof uses only the properties (1), (2), (3), and (4). It does not use any other special property of Lebesgue measure. It follows that there is no *real-valued* measure function μ that satisfies *all four properties* on *all subsets* of real numbers. In particular, suppose that $\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$ for arbitrary disjoint subsets A and B of $\mathbb{R}$. Using the notation introduced in the above proof, this would imply that

$$3 \geq \mu^*\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} \mu^*(E_i) = n\mu^*(E)$$

for all integers $n \geq 1$. The only way for this to occur (for real $\mu^*(E) \geq 0$) is for $\mu^*(E) = 0$; but then E would be a measurable set, a contradiction. However, the equality $\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$ is valid for all disjoint *measurable* sets; and this is the reason for restricting μ to the family $\mathcal{M}$ of all measurable subsets of $\mathbb{R}$.

A family $\mathcal{A}$ of sets is called an **algebra** if $\emptyset \in \mathcal{A}$, $E \in \mathcal{A} \Rightarrow \complement E \in \mathcal{A}$, and $\mathcal{A}$ is closed under finite unions (and hence also under finite intersections). An algebra of sets is called a σ-**algebra** if it is also closed under *countable* unions and intersections. The family $2^{\mathbb{R}}$ of all subsets of $\mathbb{R}$ is a σ-algebra. By the above theorems, the family $\mathcal{M}$ of all Lebesgue measurable subsets of $\mathbb{R}$ is also a σ-algebra. It is clear that an arbitrary intersection of σ-algebras is again a σ-algebra. Let $\mathcal{B}$ denote the σ-algebra that is the intersection of all σ-algebras that contain the open sets. This σ-algebra $\mathcal{B}$, called the family of **Borel sets**, is the *smallest* σ-algebra that contains all the open sets. All countable sets, all intervals, all closed sets, all open sets, all G_δ's, and all F_σ's are Borel sets. Every Borel set is measurable, but there are many measurable sets that are *not* Borel sets! Thus $\mathcal{B} \subsetneq \mathcal{M} \subsetneq 2^{\mathbb{R}}$. See [7] for proofs and much more, including the Integrals of Lebesgue, Denjoy, Perron, and Henstock simply and systematically presented. It turns out that Henstock = Perron = Denjoy $\supset$ Lebesgue $\supset$ Riemann; and Henstock's is the simplest [8].

References

Riemann [1826–1866] • Baire [1874–1932] • Lebesgue [1875–1941] • Borel [1871–1956]

1. Asplund & Bungart, A First Course in Integration, Holt, Rinehart & Winston 1966.
2. R. Baire, Sur les fonctions de variables réelles, *Ann. di Mat.* (3), **III** (1899), 1–123. Contains, among other things, the "Osgood-Baire Category Theorem" on $\mathbb{R}^n$.
3. Ralph P. Boas, Jr., A Primer of Real Functions, *Carus Mathematical Monographs,* NUMBER THIRTEEN, Math. Assoc. Amer. 2nd edition 1972.
4. N. Bourbaki, *Elements of Mathematics*, **Integration**, Hermann, Paris & Addison-Wesley, Reading, Mass. 1965.
5. Alain Connes, Noncommutative Geometry, Academic Press, 1990. Monumental!
6. G. Choquet, Lectures on Analysis, 3 vols, W. A. Benjamin 1969. Vol. 1: Baire spaces; Capacities & Radon measures; Topological vector spaces. Vol. 2: Extreme points of convex sets; The Minkowski-Krein-Milmann Theorem; Bochner's Theorem; Choquet boundary. Vol. 3: Conical, Affine, Cylinder, and Wiener measures.
7. R. Gordon, The Integrals of Lebesgue, Denjoy, Perron, and Henstock, *Graduate Studies in Mathematics*, vol. **4**, Amer. Math. Soc. 1994. A unified & fresh exposition.
8. R. McLeod, The Generalized Riemann Integral, *Carus Mathematical Monographs,* NUMBER TWENTY, Math. Assoc. Amer. 1980. A clear, detailed, and complete exposition for beginners (or anyone) of the **Henstock Integral**, with examples & exercises. The **Henstock Integral** is a direct and simple variation of Riemann's approach to integration, which results in an integral more general than Lebesgue's with all its power, plus the best **fundamental theorem of calculus**. No previous knowledge of either the Riemann or Lebesgue integral is needed to read and study this book. This should be required reading for every serious student of mathematics.
9. I. P. Natanson, Theory of Functions of a Real Variable, 2 vols., English translation from the Russian by Leo F. Boron & Edwin Hewitt, Frederick Ungar, New York 1955.
10. Riesz & Nagy, Functional Analysis. The 1990 Dover reprint of the English edition published by Frederick Ungar in 1955 contains an appendix on *Linear Transformations which extend beyond Hilbert space*. This book is famous for its beautiful exposition of beautiful & important mathematics. Frigyes Riesz [1880–1956] pioneered functional analysis.
11. Stanisław Saks, Theory of the Integral, Hafner Pub. Co., New York 1937. A classic.
12. Laurent Schwartz, Radon Measures on Arbitrary Topological Spaces and Cylindrical Measures, Oxford University Press 1973. With applications to Probability Theory, Gauss measures & Brownian motion.
13. ANDRÉ WEIL, L'INTÉGRATION DANS LES GROUPES TOPOLOGIQUES ET SES APPLICATIONS, deuxième édition, HERMANN & C^{ie} ÉDITEURS, PARIS 1953. A definitive classic.