Statistical methods for financial models driven by Lévy processes

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Program

I. Background on Lévy processes

II. Introduction to financial models driven by Lévy processes

III. Classical statistical methods

IV. Recent nonparametric methods based on low- and high-frequency sampling

Part II: Introduction to financial models driven by Lévy processes

Modeling of historical asset prices

Problem: “Construct” stochastic processes that account for the known features of stock prices dynamics.

Motivations: Sensible allocation of money in a portfolio of assets. Risk assessment.

What has been done?

- Geometric Brownian Motion
- Lévy based modeling
- Stochastic volatility models

Geometric Brownian Motion

- The model: *The "return" of the stock during a small time span dt is approx. normally distributed with constant mean and variance*:

$$\frac{S_{t+dt} - S_t}{S_t} \approx \underbrace{\mu}_{\text{Mean return}} dt + \underbrace{\sigma}_{\text{Volatility}} d \underbrace{W_t}_{\text{B.M.}}.$$

More precisely,

$$\underbrace{\log \frac{S_{t+\Delta t}}{S_t}}_{\text{Log Return on } [t,t+\Delta t)} = \underbrace{(\mu - \frac{\sigma^2}{2})}_{b} \Delta t + \sigma \underbrace{(W_{t+\Delta t} - W_t)}_{\sim N(0,\Delta t)}.$$

Equivalently,

$$S_t = S_0 \exp\{bt + \sigma W_t\}, \quad t \geq 0,$$

where $\{W_t\}$ is a Wiener process.

- Implications:
  - *Efficiency*:

    Future prices depends on the past only through the present value (Markov property).
  - Log returns in disjoint periods are independent and Normally distributed:

$$R_1^{\Delta} := \log \frac{S_{\Delta}}{S_0}, \ldots, R_n^{\Delta} := \log \frac{S_{n\Delta}}{S_{(n-1)\Delta}} \quad \stackrel{\text{i.i.d.}}{\sim} \quad \mathcal{N}(b\Delta, \sigma^2\Delta).$$

  - Continuously varying stock prices or, equivalently, continuous flow of information in the market.

- Empirical evidence:
  - The distribution of returns exhibit *heavy tails* and *high kurtosis*.
- Natural questions:
  - Can we construct a model that allows fat-tail marginal distributions, while preserving the statistical qualities of the increments and continuity? No!!
- A possible solution: Allow jumps in the process while preserving all statistical properties of the increments of a Brownian motion:

$$\Longrightarrow \text{Lévy Processes} \Longleftarrow$$

- Why jumps?
  - The prices moves discontinuously driven by discrete trades
  - "Sudden large" changes due to arrival of information

Geometric Lévy Motion

- The Model:

$$\underbrace{\log \frac{S_{t+\Delta t}}{S_t}}_{\text{Log Return on } [t,t+\Delta t)} = \underbrace{X_{t+\Delta t} - X_t}_{\text{Increment of a Levy Process}} \iff S_t = S_0 e^{X_t}$$

- Implications:

1. Equally-spaced Log returns

$$R_i := \log \frac{S_{i\Delta t}}{S_{(i-1)\Delta t}} = X_{i\Delta t} - X_{(i-1)\Delta t},$$

are independent and identically distributed with law $\mathcal{L}(X_{\Delta t})$.

2. $\mathbb{E}X_t = mt$ and $\mathrm{Var}X_t = \sigma^2 t$.

Pitfalls of Geometric Lévy models

Empirical evidence: [Cont: 2001]

- Volatility clustering: High-volatility events tend to cluster in time
- Leverage phenomenon: volatility is negatively correlated with returns
- Some sort of long-range memory: Returns do not exhibit significant autocorrelation; however, the autocorrelation of *absolute returns* decays slowly as a function of the time lag.

Conclusion: “Need” for increasingly more complex models

Other issues:

- Measurement of volatility?
- Measurement of dependence or correlation?

Other Lévy-based alternatives

Time-changed Lévy process: [Carr, Madan, Geman, Yor etc.]

$$\log S_t/S_0 = X_{T_t},$$

T_t is an increasing random process (Random Clock).

Stochastic volatility driven by Lévy processes: [B-N and Shephard]

$$\log S_t/S_0 = \int_0^t (\mu - \frac{\sigma_t^2}{2})dt + \int_0^t \sigma_t dW_t.$$
$$d\sigma_t^2 = -\lambda\sigma_t^2 dt + dX_{\lambda t},$$

$\{X_t\}_{t\geq 0}$ is a Lévy process that is nondecreasing.

Stochastic volatility with jumps in the return:

$$\log S_t/S_0 = \int_0^t \mu_u du + \int_0^t \sigma_u dW_u + \begin{cases} X_t \\ \sum_{u\le t} h(\Delta X_u, u) \end{cases},$$

$\Delta X_t =$ Size of the jump of X at time t, and $h(0,\cdot) = 0$.

SDE with jumps in the returns and the volatility: [Todorov 2005]

$$\log S_t/S_0 = \mu t + + \int_0^t \sigma_{u-} dW_u + \sum_{u\le t} h(\Delta X_u),$$

$$\sigma_t^2 = \sum_{u\le t} f(t-u)k(\Delta X_u),$$

$h(0) = k(0) = 0$

Summary

1. Exponential Lévy models are some of the simplest and most practical alternatives to the shortfalls of the geometric Brownian motion.

2. Capture several stylized empirical features of historical returns.

3. Limitations: Lack of stochastic volatility, leverage, quasi-long-memory, etc.

4. Lévy processes have been increasingly becoming an important tool in asset price modeling.