Statistical methods for financial models driven by Lévy processes

Part III: Traditional Parametric Methods for Geometric Lévy Model

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Outline

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Set-up

1. S_t the price of an asset at time t;
2. S_t is assumed to follow a Geometric Lévy model:

$$S_t = S_0 e^{X_t}, \qquad \text{where } X_t \text{ is a Lévy Process.}$$

3. Data consists of observations of the process at equally-spaced times:

$$S_0, S_\delta, \ldots, S_{n\delta};$$

4. We assume a parametric model for X:

$$\theta = (\theta_1, \ldots, \theta_k) \in \mathbb{R}^k.$$

Outline

Problems of interest

1. Estimate the true (unknown) parameter $\bar{\theta}$ based on the sample observations:

$$\hat{\theta} = \hat{\theta}(S_0, \ldots, S_{n\delta});$$

2. Analyze the behavior of the estimators as a function of the sampling time mesh δ from a practical and theoretical point of view:
   - Is there "consistency" of the estimation for different δ?
   - Do we expect better estimation performance when $\delta \to 0$ (even if disregard some "Microstructure noise effect")?
   - What are expected estimation performance under certain microstructure effects?

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Preliminary Remarks

1. Note that statistical estimation based on $S_0, S_\delta, \ldots, S_{n\delta}$ is equivalent to statistical estimation based on the increments of the process X (or the log returns):

$$R_1^\delta = \log \frac{S_\delta}{S_0} = X_\delta - X_0, \ \ldots \ , R_n^\delta = \log \frac{S_{n\delta}}{S_{(n-1)\delta}} = X_{n\delta} - X_{(n-1)\delta};$$

2. Under a Geometric Lévy Model, the increments can be thought of as independent draws from common population distribution F_δ;

3. We assume from now on that the population distribution is (absolutely) continuous governed by a probability density function $f_\delta(\cdot) = f_\delta(\cdot;\theta)$:

$$P(a < R_i^\delta < b) = \int_a^b f_\delta(x)dx;$$

Maximum Likelihood Estimation (MLE)

1. Maximum Likelihood Principle:

   Most sensible values of the parameters are those that maximize the likelihood (chance) of observing the sample data.

2. General implementation method:
   - $r_1^\delta, \ldots, r_n^\delta$ are the sample observations of n equally-spaced returns (with time-span δ).
   - Compute the Likelihood function defined by

$$L_\delta(\theta; r_1, \ldots, r_n) = \Pi_{i=1}^n f_\delta(r_i; \theta).$$

   - The *Maximum Likelihood Estimate* (when it exists) is defined by

$$\hat{\theta} = \hat{\theta}^\delta(r_1, \ldots, r_n) = \mathrm{argmax}_\theta \, L_\delta(\theta; r_1, \ldots, r_n).$$

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Implementation issues

1. Lévy-based models are often described in terms of their Lévy density. As a consequence, the characteristic function or Fourier transform $\hat{f}_\delta$ is known in a closed form, but the density f_δ is unknown or "intractable".
2. A possible solutions:
   - Inversion formula.

$$f_\delta(r\,;\,\theta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-izr}\hat{f}_\delta(z\,;\,\theta)\,dz.$$

   - Coupled with an approximate of the integral via Fast Fourier Transform.

Method of Moment Estimators

1. General idea:

   Choose the parameter values that match the theoretical moments with the sample empirical moments.

2. General implementation method:
   - $r_1^\delta, \ldots, r_n^\delta$ are the sample observations of n equally-spaced returns (with time-span δ).
   - Compute the theoretical and sample (centered) moments (one for each estimator)

   $$m_k^\delta(\theta) = \mathbb{E}[(R_i^\delta)^k], \quad \hat{m}_k = \frac{1}{n}\sum_{i=1}^{n}(r_i^\delta)^k].$$

   - The *Method of Moment Estimate* (when it exists) is defined by solutions $\hat{\theta}$ of the system of equations:

   $$m_k^\delta(\hat{\theta}) = \hat{m}_k, \quad k = 1, \ldots, d..$$

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1. The model
2. Problem formulation
3. Statistical Methodology
4. Case study 1: Variance Gamma Process
    - Finite-sample performance by simulations
    - Some preliminary empirical results
5. Case study 2: Normal Inverse Gaussian
    - Finite-sample performance by simulations

Variance Gamma Model

- Definition: (Madan et. al. 98)

$$X_t = \sigma W_{\tau_t} + \theta \tau_t + b t,$$

W standard Brownian motion and τ_t Gamma Lévy process; that is, a Lévy process such that $\tau_1 \stackrel{D}{\sim} \mathrm{Gamma}\,(\alpha = \frac{1}{\kappa}, \beta = \kappa)$ (in particular, $\mathbb{E}\tau_t = t$ and $\mathrm{Var}(\tau_t) = \nu t$).

- Moments:

$$\mu_1^\delta := \mathbb{E}(X_\delta) = (\theta + b)\delta,$$

$$\mu_2^\delta := \mathrm{Var}(X_\delta) = (\sigma^2 + \theta^2\kappa)\delta,$$

$$\mu_3^\delta := \mathbb{E}(X_\delta - \mathbb{E}X_\delta)^3 = (3\sigma^2\theta\kappa + 2\theta^3\kappa^2)\delta,$$

$$\mu_4^\delta := \mathbb{E}(X_\delta - \mathbb{E}X_\delta)^4$$

$$= (3\sigma^4\kappa + 12\sigma^2\theta^2\kappa^2 + 6\theta^4\kappa^3)\delta + 3(\mu_2^\delta)^2.$$

Variance Gamma Model

- Some Interpretation for θ is small:

$$\mathrm{Var}(X_\delta) \approx \sigma^2\delta, \quad \mathrm{Kurt}(X_\delta) := \frac{\mu_4}{\mu_2^2} - 3 \approx \frac{3\kappa}{\delta}.$$

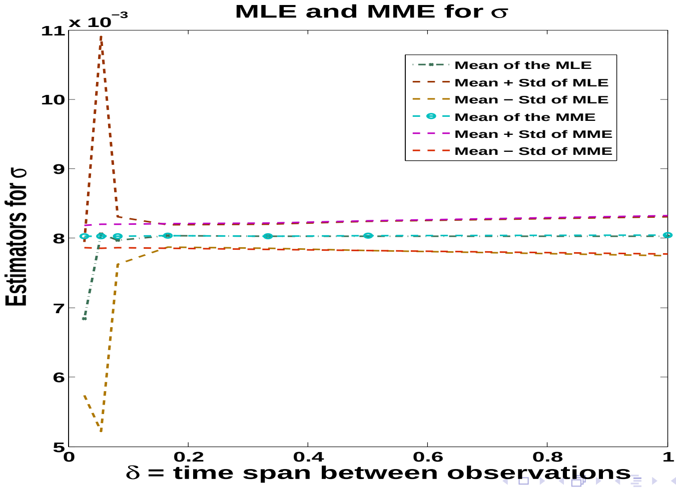
- Density has a “closed form" in terms of “Bessel functions of second kind”:

$$f_\delta(x) = \frac{2e^{\theta(x-b\delta)/\sigma^2}}{\sigma\sqrt{2\pi}\kappa^{\delta/\kappa}\Gamma(\frac{\delta}{\kappa})}\left(\frac{|x-b\delta|}{\sqrt{\frac{2\sigma^2}{\kappa}+\theta^2}}\right)^{\frac{\delta}{\kappa}-\frac{1}{2}}$$

$$\times K_{\frac{\delta}{\kappa}-\frac{1}{2}}\left(\frac{|x-b\delta|\sqrt{\frac{2\sigma^2}{\kappa}+\theta^2}}{\sigma^2}\right),$$

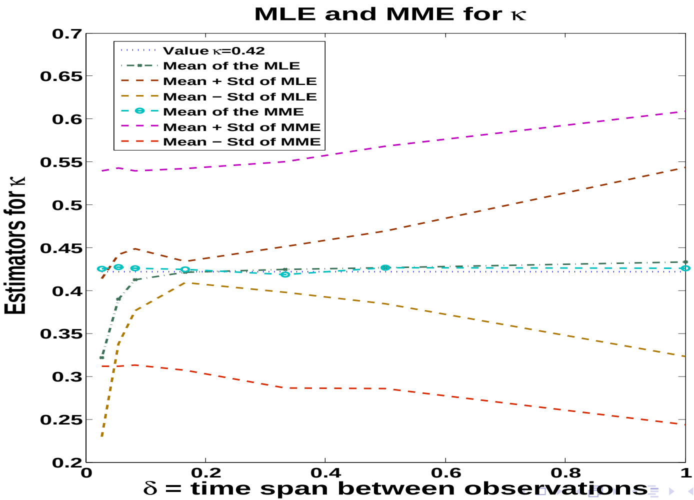
MME and MLE for σ in the VG model

True Value=0.0080; Sampling time mesh=δ=1/36,1/18,1/12,1/6,1/3,1/2,1



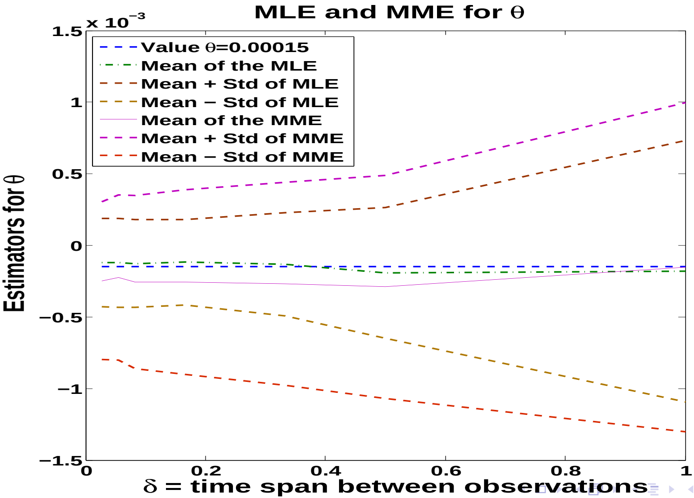
MME and MLE for κ in the VG model

True Value=0.422; Sampling time mesh=δ=1/36,1/18,1/12,1/6,1/3,1/2,1



MME and MLE for θ in the VG model

True Value=-0.00015; Sampling time mesh=δ=1/36,1/18,1/12,1/6,1/3,1/2,1



MLE and MME for INTC Stock Data 2005

1/36	1/18	1/12	1/6	1/3	1/2	1
0.019285	0.049581	0.069021	0.166220	0.234176	0.209791	0.287256
0.012048	0.012007	0.011804	0.012017	0.012305	0.012452	0.012650
0.000319	-0.000335	-0.000160	0.001937	0.001985	0.001952	0.001338
-0.000690	-0.000039	-0.000212	-0.002317	-0.002353	-0.002323	-0.001709
0.033768	0.065000	0.098626	0.183866	0.269412	0.157855	0.138268
0.012173	0.012038	0.011949	0.011978	0.012342	0.012389	0.012541
0.002115	0.001012	0.001328	0.003228	0.002413	0.002779	0.004111
-0.002486	-0.001383	-0.001700	-0.003599	-0.002784	-0.003151	-0.004483

MLE and MME for INTC Stock Data 2005

5s	10s	15s	20s	30s	1m	2m	3m	5m	6m
0.000577	0.000963	0.001619	0.002321	0.005241	0.008061	0.042436	0.010300	0.015306	0.018678
0.019600	0.016942	0.015773	0.015152	0.014509	0.013763	0.013302	0.012848	0.012454	0.012315
-0.001525	-0.000083	-0.000817	0.001442	0.002536	-0.004006	-0.003547	0.000469	-0.001286	0.000453
0.001154	-0.000288	0.000445	-0.001813	-0.002907	0.003634	0.003175	-0.000841	0.000914	-0.000825
0.000577	0.000963	0.001619	0.002321	0.005240	0.008050	0.042180	0.010300	0.015301	0.018677
0.019600	0.016942	0.015773	0.015151	0.014508	0.013758	0.013282	0.012848	0.012453	0.012315
-0.001525	-0.000083	-0.000817	0.001442	0.002537	-0.004012	-0.003572	0.000469	-0.001286	0.000454
0.001154	-0.000288	0.000445	-0.001813	-0.002908	0.003641	0.003200	-0.000841	0.000915	-0.000825

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Normal Inverse Gaussian

- Definition: (Madan et. al. 98)

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W standard Brownian motion and τ_t is an Inverse Gaussian subordinator such that $\mathbb{E}\tau_t = t$ and $\mathrm{Var}(\tau_t) = \nu t$.

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$$= (3\sigma^4\kappa + 18\sigma^2\theta^2\kappa^2 + 15\theta^4\kappa^3)\delta + 3(\mu_2^\delta)^2.$$

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- Density has a “closed form" in terms of “Bessel functions of second kind”:

$$f_\delta(x) = \frac{Ce^{\theta(x-b\delta)/\sigma^2}}{\sqrt{(x-b\delta)^2+\delta^2\sigma^2/\kappa}} K_1\left(B\sqrt{(x-b\delta)^2+\delta^2\sigma^2/\kappa}\right),$$

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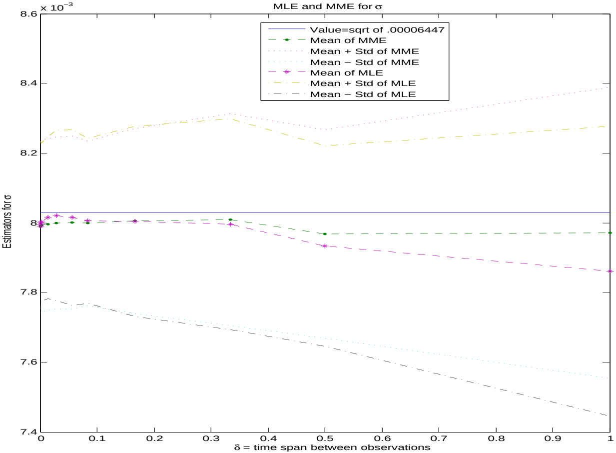
True Value=0.0080; $\delta = 1, 1/2, 1/3, 1/6, 1/12, ..., 1/(6 * 60 * 12)$



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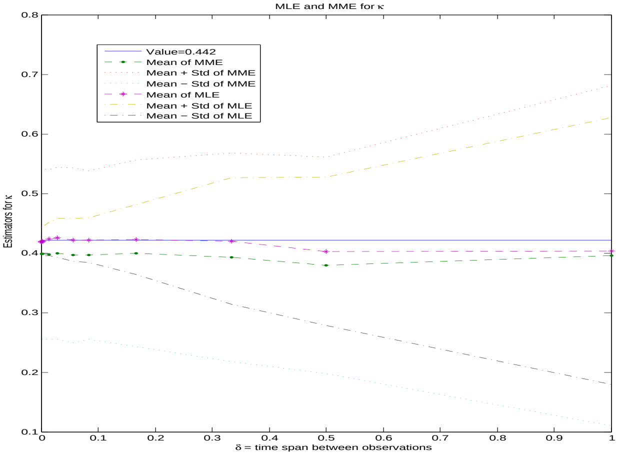

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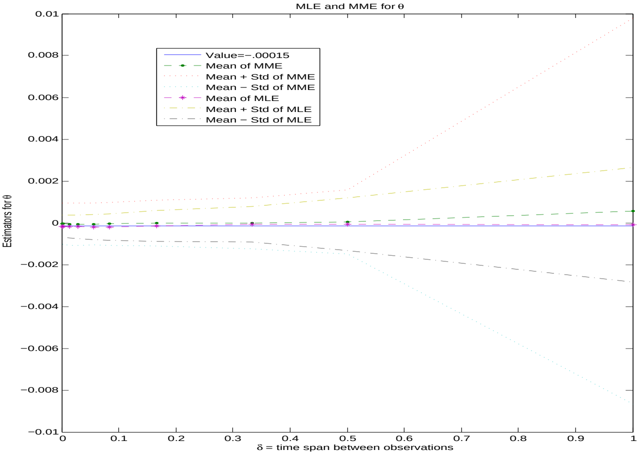

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