de Finetti's Control Problem

Víctor Rivero [1]

Centro de Investigación en Matemáticas, México

[1] based on joint work with Andreas E. Kyprianou and Renming Song

- In 1903 Filip Lundberg realized that Poisson processes lies at the heart of non-life insurance models.
- His "discovery" is similar to the recognition by Bachelier in 1900 that Brownian motion is the key building block for financial models.
- Later, around 1930, Harald Cramér and its collaborators incorporate Lundberg's ideas in the emerging theory of stochastic processes. In doing so Cramér contributed considerably to laying the foundation of both non-life insurance mathematics as well a probability theory. The basic model coming out from this contributions is the so-called *Cramér-Lundberg model*.

Definition (Cramér-Lundberg model)

- *Claim sizes* are $(X_k)_{k\geq 1}$ are positive i.i.d. random variables having common distribution F, and finite mean.
- *Claim times*, the claims occur at random instants

$$0 < T_1 < T_2 < \cdots < T_k < \dots$$

- *Claim arrival process*, the number of claims in the interval $[0,t]$ is denoted by

$$N(t) = \sup\{n \geq 1 : T_n \leq t\}, \qquad t \geq 0;$$

  $\sup\{\emptyset\} = 0$.
- *The inter-arrival times*,

$$Y_1 = T_1, Y_k = T_k - T_{k-1}, \qquad k \geq 2,$$

  are i.i.d. exponentially distributed random variables with mean $\mathbb{E}(Y_1) = 1/\lambda$.

We define

- the *total claim amount process* $\{S_t, t \geq 0\}$,

$$S_t = \begin{cases} \sum_{i=1}^{N(t)} X_i, & \text{if } N(t) > 0 \\ 0, & \text{if } N(t) = 0. \end{cases}, \qquad t \geq 0.$$

- the associated *risk process*

$$U_t = u + ct - S_t, \qquad t \geq 0,$$

  where u denotes the *initial capital* and $c > 0$ stands for the *premium income rate*.

- the *probability of ruin* before time T is

$$\varphi(u, T) = \mathbb{P}(U_t \leq 0, \text{ for some } t \leq T | U_0 = u), \qquad u \geq 0,$$

- the *probability of ruin*

$$\varphi(u) = \mathbb{P}(U_t \leq 0, \text{ for some } t < \infty | U_0 = u), \qquad u \geq 0,$$

Some well known facts

- The premium income rate c, is chosen such that the *net profit condition* is satisfied

$$\mathbb{E}(U_1) - u = c - \lambda\,\mathbb{E}(X_1) > 0.$$

Which ensures that with strictly positive probability the insurance company will not go to bankrupt, i.e.

$$1 - \varphi(u) = \mathbb{P}(U_t > 0, \text{ for all } t < \infty | U_0 = u) > 0.$$

- In this case, a consequence of the SLLN is

$$\lim_{t\to\infty} \frac{U(t)}{t} = c - \lambda\,\mathbb{E}(X_1), \qquad a.s.$$

and hence $\{U_t, t \geq 0\}$ drifts towards ∞,

$$\lim_{t\to\infty} U(t) = \infty, \qquad \text{a.s.}$$

- The *Pollaczek-Khintchine* formula establishes that

$$1 - \varphi(u) = \frac{\rho}{1+\rho} \sum_{0}^{\infty} \frac{1}{(1+\rho)^n} F_I^{n*}(u),$$

where $\rho := \frac{c}{\lambda\mu} - 1 > 0$ and

$$F_I(x) := \frac{1}{\mathbb{E}(X_1)} \int_0^x (1 - F(y))\,dy, \qquad x \geq 0.$$

Theorem (Cramér Lundberg Theorem)

Assume that $c - \lambda\,\mathbb{E}(X_1) > 0$, *and that there exists a* $\theta > 0$, *such that*

$$\int_0^\infty e^{\theta x}\overline{F}(x)\,dx = \frac{c}{\lambda}.$$

The index θ *is the so-called Lundberg exponent or adjustment coefficient. Then*

- *For all initial capital* $u \geq 0$

$$\varphi(u) = \mathbb{P}(U_t \leq 0,\ \textit{for some } t < \infty | U_0 = u) \leq e^{-\theta u}.$$

- *If, moreover,* $\int_0^\infty x e^{\theta x}\overline{F}(x)\,dx < \infty$, *then*

$$\lim_{u\to\infty} e^{\theta u}\varphi(u) = \frac{1}{\frac{\theta}{\rho\,\mathbb{E}(X_1)}\int_0^\infty x e^{\theta x}(1 - F(x))\,dx} < \infty$$

- With the intention of making the study of ruin under the Cramér-Lundberg dynamics more realistic, in 1957, de Finetti suggest the possibility that *dividends are paid out to share holders up to the moment of ruin. Further, the payment of dividends should be made in such a way as to optimize the expected net present value of the total dividends paid to the shareholders from time zero until ruin.*

It is natural to make the following assumptions on the cumulated dividend payments up to time t, $\{L_t, t \geq 0\}$

(i) ruin does not occur due to dividend payments;

(ii) $L_0 = 0$ and the paths of L are non-decreasing;

(iii) payments have to stop after the event of ruin;

(iv) decisions have to be fixed in a predictable way.

Mathematical formulation of de Finetti's control problem

- Let $\xi = \{L_t^\xi : t \geq 0\}$ be a *dividend strategy*, i.e. a left-continuous non-negative non-decreasing process, adapted to the (completed and right continuous) filtration $\{\mathcal{F}_t : t \geq 0\}$ of U.

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- The *controlled risk process* w.r.t. dividend strategy ξ, is thus $X^\xi = \{X_t^\xi : t \geq 0\}$ where

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- Let $\sigma^\xi = \inf\{t > 0 : X_t^\xi < 0\}$ be the ruin time when the dividend payments are taken into account.
- A dividend strategy is called *admissible* if at any time before ruin a lump sum dividend payment is smaller than the size of the available reserves; in other words $L_{t+}^\xi - L_t^\xi \leq \max\{X_t^\xi, 0\}$ for $t \leq \sigma^\xi$.

Denoting the set of all admissible strategies by Ξ, the expected value discounted at rate $q > 0$ of the dividend policy $\xi \in \Xi$ with initial capital $x \geq 0$ is given by

$$v_\xi(x) = \mathbb{E}_x \left(\int_{[0,\sigma^\xi]} e^{-qt} \, dL_t^\xi \right),$$

where $\mathbb{E}_x$ denotes expectation with respect to $\mathbb{P}_x$ and $q > 0$ is a fixed rate.

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de Finetti's dividend problem consists of solving the following stochastic control problem: characterize

$$v^*(x) := \sup_{\xi \in \Xi} v_\xi(x) \tag{1}$$

and, further, if it exists, establish a strategy ξ^* such that $v^*(x) = v_{\xi^*}(x)$.

Two examples of dividend strategies: Threshold strategy

Such strategy pays out dividends continuously at a rate a whenever the current reserve U is above level b, i.e.

$$L_t = a \int_0^{t \wedge \sigma^\xi} 1_{\{U_{s-} \geq b\}} ds, \qquad t \geq 0.$$

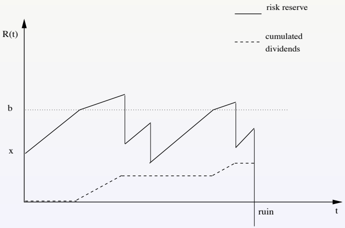

Figure 1. A sample path of the Cramér-Lundberg model under a control of *threshold* type

Two examples of dividend strategies: Barrier strategy

For a given $b \geq 0$, such a strategy pays out all the reserve above b immediately at $t = 0+$ (representing a singular component in the strategy) and subsequently all incoming premiums that lead to a surplus above b are immediately distributed as dividends,

$$L_t^{\xi} = (x-b)1_{\{x>b\}} + \int_0^{t\wedge\sigma^{\xi}} 1_{\{X_{s-}^{\zeta}=b\}} ds = b \vee (\sup_{s\leq t} U_s) - b$$

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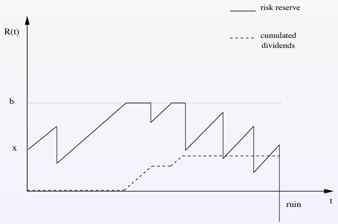

Figure 2. A sample path of the Cramér-Lundberg model under a control of *barrier* type

Other examples of dividend strategies are: *Band strategies* and *impulse strategy.* Bühlmann, H. (1970) proposes other types of strategies.

- The barrier strategy is an intuitively natural strategy for profit participation in the risk process.
- It was first proposed by de Finetti in 1957 and he showed that a certain barrier strategy maximizes expected discounted dividend payments if the underlying risk reserve process is modeled as a simple random walk.
- In 1969, Gerber proved that for the particular case of exponentially distributed claim amounts, the barrier strategy is optimal.
- Gerber also proved that for an appropriate choice of jump distribution, the above described barrier strategy is not optimal.
- What are sufficient conditions for a barrier strategy to be optimal and what is the value of the optimal level?

Spectrally negative Lévy risk processes

The risk process started from $u = 0$

$$\{U_t = ct - S_t,\ t \geq 0\}$$

(a) has right continuous left-limited paths,

(b) has independent increments, i.e. for any $0 \leq t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_n < \infty$ the r.v.

$$(U_{t_1} - U_{t_0}, U_{t_2} - U_{t_1}, \ldots, U_{t_n} - U_{t_{n-1}}),$$

are independent.

(c) has stationary increments $U_{t+s} - U_t \overset{\text{Law}}{=} U_s$

(d) has no-positive jumps, $\Delta U_t = U_t - U_{t-} \leq 0$, $t \geq 0$, a.s.

Said otherwise, U is a *spectrally negative Lévy process [SNLP]*.

Preliminaries on SNLP

- $U = \{U_t : t \geq 0\}$ with probabilities $\{\mathbb{P}_x : x \in \mathbb{R}\}$ will always denote a spectrally negative Lévy process [SNLP] (i.e. $\Pi_U(0, \infty) = 0$ and $-U$ is not a subordinator), s.t. net profit condition $\mathbb{E}(U_1) \geq 0$ holds.

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- For $\theta \geq 0$ we may work with the Laplace exponent

$$\psi(\theta) := \log \mathbb{E}_0(e^{\theta U_1}) = a\theta + \frac{1}{2}\sigma^2\theta^2 + \int_{(0,\infty)} (e^{\theta x} - 1 - \theta x 1_{\{x > -1\}})\Pi_U(dx)$$

$a, \sigma \in \mathbb{R}$, $\int_{(-\infty,0)} (1 \wedge x^2)\Pi_U(dx) < \infty$; which satisfies that it is strictly convex, $\psi(0) = 0$, $\psi(\infty) = \infty$ and $\mathbb{E}_0(U_1) = \psi'(0+)$.

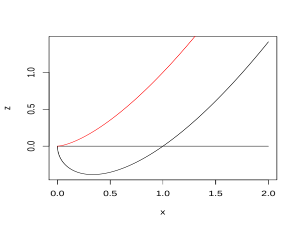

Figure: Typical shape of ψ. Black $\psi'(0+) < 0$, Red $\psi'(0+) \geq 0$.

Scale functions

For each $q \geq 0$, the, so-called, q-scale function $W^{(q)} : \mathbb{R} \mapsto [0, \infty)$ is defined by $W^{(q)}(x) = 0$ for $x < 0$ and elsewhere continuous and increasing satisfying

$$\int_0^\infty e^{-\beta x} W^{(q)}(x) dx = \frac{1}{\psi(\beta) - q}$$

for all β sufficiently large ($\psi(\beta) > q$).

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Let $\tau_a^- = \inf\{t > 0 : U_t < a\}$, $\tau_b^+ = \inf\{t > 0 : U_t > b\}$, $a, b \in \mathbb{R}$. We have the classical identity

$$\mathbb{E}_x(e^{-q\tau_a^+}\mathbf{1}_{(\tau_a^+ < \tau_0^-)}) = \frac{W^{(q)}(x)}{W^{(q)}(a)}$$

for $q \geq 0$, $0 \leq x \leq a$.

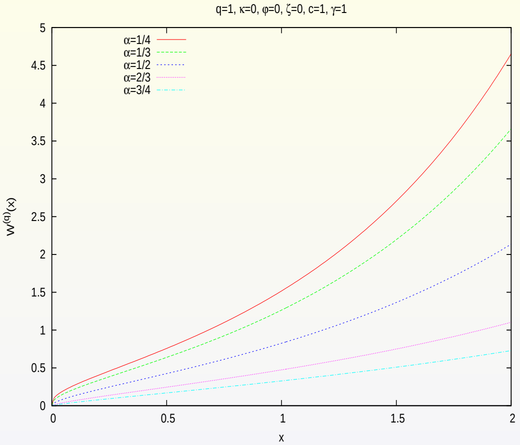

Figure: Scale functions $W^{(q)}(x)$ for a Generalized Tempered Stable process which oscillates: concavity/convexity.

$\frac{\psi(\lambda)}{\lambda} = \phi(\lambda) = \kappa + \zeta\lambda + c((\lambda+\gamma)^{\alpha} - \gamma^{\alpha}),\ \lambda > 0;\ \kappa \geq 0,\ \zeta \geq 0,\ c > 0,$
$\alpha \in (0,1)$.

In 2007, Avram, Palmowski and Pistorius considered de Finetti's control problem in the framework of Spectrally negative Lévy risk processes. They expressed the cumulative dividends process, when the barrier strategy at level a is chosen, in the form

$$L_t^a = a \vee \overline{U}_t - a$$

for some $a \geq 0$ where $\overline{U}_t := \sup_{s \leq t} U_s$.
In that case the controlled process $X_t^a = U_t - L_t^a$ is a spectrally negative Lévy process reflected in the barrier a.

Avram et al. proved that for any $a > 0$, the expected value discounted at rate $q > 0$ of the barrier strategy at level a is given by

$$v_a(x) := \mathbb{E}_x \left(\int_{[0,\sigma^a]} e^{-qt} dL_t^a \right)$$
$$= \begin{cases} W^{(q)}(x)/W^{(q)\prime}(a), & -\infty < x \leq a, \\ x - a + W^{(q)}(a)/W^{(q)\prime}(a), & \infty > x > a. \end{cases}$$

where $\sigma^a = \inf\{t > 0 : U_t^a < 0\}$. Kyprianou and Palmowski (2007) proved that

$$v_a(x) := \mathbb{E}_x \left(\left(\int_{[0,\sigma^a]} e^{-qt} dL_t^a \right)^n \right) = n! \frac{W^{(qn)}(a)}{W^{(qn)}(a)} \prod_{k=1}^{n} \frac{W^{(qk)}(a)}{W^{(qk)\prime}(a)}$$

- Avram et al. gave sufficient conditions for the optimal strategy to consist of a simple barrier strategy. This sufficient conditions are phrased in terms of a variational inequality involving the value of a barrier strategy which itself can be expressed in terms of the associated scale function $W^{(q)}$.
- After Avram et al. Loeffen, in 2007, made a decisive statement connecting the shape of the scale function $W^{(q)}$ to the existence of an optimal barrier strategy.

Theorem (Loeffen (2007))

Suppose that U is such that its scale functions are sufficiently smooth, meaning that $W^{(q)}$ is in $C^1(0,\infty)$ if U is of bounded variation and $W^{(q)}$ is in $C^2(0,\infty)$ otherwise. Let

$$a^* = \sup\{a \geq 0 : W^{(q)\prime}(a) \leq W^{(q)\prime}(x) \text{ for all } x \geq 0\},$$

(which is necessarily finite) where we understand $W^{(q)\prime}(0) = W^{(q)\prime}(0+)$. Then the barrier strategy at a^* is an optimal strategy if

$$W^{(q)\prime}(a) \leq W^{(q)\prime}(b) \text{ for all } a^* \leq a \leq b < \infty.^{a}$$

[a] function $W^{(q)}$ is convex beyond the global minimum of its first derivative.
[b] π is completely monotone if $\pi \in C^\infty(0,\infty)$ and $(-1)^n \pi^{(n)} \geq 0$.

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If the Lévy measure of $-U$, has a completely monotone density $\frac{\Pi_U(-dx)}{dx} = \pi(x)$ [b], then the barrier strategy at a^* is optimal.

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What are sufficient conditions on the Lévy measure so that the conditions of Loeffen's Theorem holds?

Theorem (Kyprianou, R. and Song (2008))

Suppose that $-U$ *has a Lévy density* $\frac{\Pi_U(-dx)}{dx} = \pi(x)$ *that is log convex*[a] *then the barrier strategy at* a^* *is optimal for de Finetti's control problem.*

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Proof based on properties of scale functions, fluctuation theory for Lévy processes and stochastic calculus for semi-martingales.

Theorem (Kyprianou, R. and Song)

If

$$\overline{\Pi}_U(-\infty,-x) := \int_{-\infty}^{-x} \pi(-s)\,ds, \quad x>0$$

is log convex, then for any $q>0$ if $\Phi(0)=0$, and $q\geq 0$ if $\Phi(0)>0$, the function $g_q(x) := e^{-\Phi(q)x}W^{(q)}(x)$, $x>0$, is concave.[a]

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Finally, if the latter assumption is satisfied and the Gaussian coefficient is strictly positive then $W^{(q)} \in C^2(0, \infty)$.

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Open problems

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- If a barrier strategy is not optimal what kind of strategy is optimal? And what are NASC on the Lévy measure for that strategy to be optimal.

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- Compound Poisson with negative exponentially distributed jumps of mean μ, arrival rate λ and positive drift c such that $\mathbb{E}(X_1) = c - \lambda/\mu > 0$.

$$W(x) = \frac{1}{c}\left(1 + \frac{\lambda}{c\mu - \lambda}(1 - e^{(\mu - c^{-1}\lambda)x})\right)$$

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- α-stable process with $\alpha \in (1, 2)$.

$$W(x) = x^{\alpha - 1}/\Gamma(\alpha).$$

Dig a little deeper ....

- Furrer (1998) studies ruin of an α-stable process with $\alpha \in (1,2)$ plus a drift ct and deduces that

$$W(x) = \frac{1}{c}(1 - E_{\alpha-1,1}(-cx^{\alpha-1}))$$

where

$$E_{\alpha-1,1}(z) = \sum_{k\geq 0} z^k/\Gamma(1+(\alpha-1)k)$$

is the two-parameter Mittag-Leffler function with indices $\alpha - 1$ and 1.

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- An unusual example from queuing theory due to Boxma and Cohen (1998). Let $\eta(x) = e^x \mathrm{erfc}(\sqrt{x})$ and consider a compound Poisson with rate λ satisfying $1-\lambda > 0$, negative jumps with d.f. $F(x,\infty) = (2x+1)\eta(x) - 2\sqrt{x/\pi}$ and unit positive drift. Then

$$W(x) = \frac{1}{1-\lambda}\left(1 - \frac{\lambda}{\nu_1 - \nu_2}(\nu_1\eta(x\nu_2^2) - \nu_2\eta(x\nu_1^2))\right).$$

where $\nu_{1,2} = 1 \pm \sqrt{\lambda}$.

- Asmussen in his book 'Ruin Probabilities' studies a compound Poisson with rate λ, negative jump of fixed size α and positive drift c. Then

$$W(x) = \frac{1}{c} \sum_{n=1}^{\lfloor x/\alpha \rfloor} e^{-\lambda(\alpha n - x)/c} \frac{1}{n!} \left(\frac{\lambda}{c}\right)^n (\alpha n - x)^n$$

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- Two new scale function occurring in study of self-similar Markov processes [Chaumont, Kyprianou and Pardo (2007)]. The Lévy processes in question have unbounded variation processes with no Gaussian component and jump measure which is stable like (with stability parameter $\alpha \in (1,2)$) near the origin and has exponentially decaying tails. Their Laplace exponents are $\Gamma(\theta+\alpha)/[\Gamma(\theta)\Gamma(\alpha)]$ and $\Gamma(\theta-1+\alpha)/[\Gamma(\theta-1)\Gamma(\alpha)]$ and the respective scale functions are

$$W(x) = (1-e^{-x})^{\alpha-1} \text{ and } W(x) = (1-e^{-x})^{\alpha-1}e^{x}.$$

New examples: preliminaries

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- The Wiener-Hopf factorization in space tells us that the Laplace exponent of H,

$$\phi(\lambda) = -\log \mathbf{E}(e^{-\lambda H_1}) = \kappa + \mathrm{d}\lambda + \int_{(0,\infty)} (1 - e^{-\lambda x}) \Pi_H(dx), \qquad \lambda \geq 0,$$

$\kappa, \mathrm{d} \geq 0$ $\int_0^\infty (1 \wedge x) \Pi_H(dx) < \infty$; is related to ψ by

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- The characteristic triple of H is given by $\kappa = \psi'(0+)$, $\mathrm{d} = \sigma^2/2$.

$$\Pi_H(x, \infty) = \int_x^\infty \Pi_X(-\infty, -y) dy, \quad x > 0$$

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- The potential measure of H is the unique measure whose Laplace transform is $1/\phi$, so

$$\int_0^\infty dt \cdot \mathbb{P}(H_t \in dx) = W(dx).$$

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- If ϕ is the Laplace exponent of your favourite subordinator the relation

$$\psi(\lambda) := \lambda\phi(\lambda), \qquad \lambda \geq 0$$

defines the Laplace exponent of a SNLP if and only if Π_H has a non-increasing density.

Parent process for given H

Theorem (Hubalek & Kyprianou 2007)

Suppose that H is a (killed) subordinator with Laplace exponent

$$\phi(\lambda) = \kappa + \mathrm{d}\lambda + \int_{(0,\infty)} (1 - e^{-\lambda x})\Pi_H(dx), \ \lambda \geq 0,$$

such that Π_H has a non-increasing density. Then there exists a spectrally negative Lévy process X, henceforth referred to as the **parent process**, *such that its associated downwards ladder height process is precisely H.*

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such that Π_H has a non-increasing density. Then there exists a spectrally negative Lévy process X, henceforth referred to as the **parent process***, such that its associated downwards ladder height process is precisely H. The Lévy triple (a, σ, Π_X) of the parent process is uniquely identified as follows.*

- *Gaussian coefficient* $\sigma = \sqrt{2\mathrm{d}}$.
- *Linear term* $\kappa = a + \int_{(-\infty,-1)} x\Pi_X(dx)$.
- *Lévy measure* $\Pi_X(-\infty, -x) = \dfrac{d\Pi_H(x)}{dx}$, $x > 0$.

Bounded and unbounded variation

- **When** $\Pi_H(0,\infty) < \infty$. The parent process is given by

$$X_t = (\kappa + \Pi_H(0,\infty))t + \sqrt{2\mathrm{d}}B_t - S_t \tag{2}$$

where $B = \{B_t : t \geq 0\}$ is a Brownian motion, $S = \{S_t : t \geq 0\}$ is an independent driftless subordinator with jump measure ν satisfying

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- **When** $\Pi_H(0,\infty) = \infty$. The parent process X always has paths of unbounded variation.

One example from the theory of pssMp

Let $c > 0$, $\nu \geq 0$ and $\theta \in (0,1)$ and ϕ be defined by

$$\phi(\lambda) = \frac{c\lambda\Gamma(\nu+\lambda)}{\Gamma(\nu+\lambda+\theta)}, \qquad \lambda \geq 0.$$

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(Γ is the usual Gamma fct.) ϕ is a Bernstein function, i.e. there exists a subordinator H with ϕ as Laplace exponent. Its characteristics are $\kappa = 0, \mathrm{d} = 0$,

$$\overline{\Pi}_H(x) := \Pi_H(x,\infty) = \frac{c}{\Gamma(\theta)} e^{-x(\nu+\theta-1)} \left(e^x - 1\right)^{\theta-1}, \qquad x > 0.$$

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$$W(x) = \frac{\Gamma(\nu+\theta)}{c\Gamma(\nu)} + \frac{\theta}{c\Gamma(1-\theta)} \int_0^x \left\{ \int_y^\infty \frac{e^{z(1-\nu)}}{(e^z-1)^{1+\theta}} dz \right\} dy, \;\; x \geq 0.$$

Two for the price of one...

$$W'(x) = \int_x^\infty \frac{e^{z(1-\nu)}}{(e^z - 1)^{1+\theta}} dz, \qquad x \geq 0$$

is non-increasing, convex and s.t. $\int_0^\infty (1 \wedge x)|W''(x)|dx < \infty$.

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$$\psi^*(\lambda) = \lambda\phi^*(\lambda) = \frac{\lambda^2}{\phi(\lambda)}, \qquad \lambda \geq 0,$$

defines the Laplace exponent of a SNLP that drifts to ∞. Its associated scale function is given by

$$W^*(x) = \frac{c}{\Gamma(\theta)} \int_0^x e^{-z(\nu+\theta-1)}(e^z-1)^{\theta-1}\,dz = \int_0^x \overline{\Pi}_H(z)\,dz, \quad x \geq 0.$$

Special and conjugate scale functions

More generally, take a **special Bernstein function**. That is to say, choose the Laplace exponent of the descending ladder height ϕ such that

$$\phi(\theta) = \kappa + \mathrm{d}\theta + \int_{(0,\infty)} (1 - e^{-\theta x})\Pi_H(dx) \quad \text{for } \theta \geq 0$$

with the assumption that Π_H is absolutely continuous with a non-increasing density and such that ϕ^* defined by

$$\phi^*(\theta) = \frac{\theta}{\phi(\theta)} \quad \text{for } \theta \geq 0,$$

is also a Bernstein function (the conjugate to ϕ) which we shall write as

$$\phi^*(\theta) = \kappa^* + \mathrm{d}^*\theta + \int_{(0,\infty)} (1 - e^{-\theta x})\Pi_{H^*}(dx).$$

This is possible if and only if the potential measure of H restricted to $(0, \infty)$ has a non-increasing density.

Special and conjugate scale functions ctd.

- Potential analysis of special Bernstein functions gives us an expression for the potential function associated to ϕ and hence an expression for the the **special scale function** whose parent process has Laplace exponent $\psi(\theta) = \theta\phi(\theta)$:

$$W(x) = \mathrm{d}^* + \kappa^* x + \int_0^x \Pi_{H^*}(y, \infty)\, dy$$

  and W is a concave function. The potential measure of H is

$$W(dx) = \mathrm{d}^* \delta_0(dx) + (\kappa^* + \Pi_{H^*}(x, \infty))\, 1_{\{x>0\}}\, dx, \;\; x \geq 0.$$

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- If it so happens that Π_H^* is absolutely continuous with non-increasing density, then we get the **conjugate scale function**

$$W^*(x) = \mathrm{d} + \kappa x + \int_0^x \Pi_H(y,\infty)dy.$$

(also concave) whose parent process has Laplace exponent $\psi^*(\theta) = \theta\phi^*(\theta)$