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Bartlett Adjustments for Empirical Goodness of Fit

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ABSTRACT: DiCiccio, Hall and Romano's computation of the third and fourth joint cumulants for the signed root of the empirical likelihood statistic is generalized to the class of empirical goodness of fit tests parametrized by λ (empirical likelihood corresponds to $\lambda = 0$). It is shown that empirical likelihood is the only member of this family which is Bartlett-correctable.

KEY WORDS: Cressie-Read statistic, empirical likelihood, goodness-of-fit tests, maximum entropy, nonparametric confidence regions.

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1 Introduction

This technical report is a reworking of the technical report of DiCiccio, Hall and Romano (1988, denoted DHR henceforth). Their notation and phrasing have been preserved wherever possible, and the argument (together with the concommittant algebra) has been expanded for greater clarity. In their technical report, DiCiccio, Hall and Romano establish that in the Edgeworth expansion for the signed root of the empirical loglikelihood ratio statistic, the third and fourth joint cumulants are of an ignorable order, so that coverage accuracy for empirical likelihood can be improved by an order of magnitude by a simple scale adjustment of the cutoff values. We replace the loglikelihood goodness-of-fit criterion with the more general Cressie-Read power-divergence goodness of fit statistic, which is indexed by the parameter λ (empirical likelihood corresponds to the special case $\lambda = 0$). We show that the Bartlett correctability of empirical likelihood is a fragile property, in that the generalized joint third and fourth cumulants include nonignorable terms directly proportional to λ.

2 Derivation of the Bartlett Adjustment

Step 1. Empirical Goodness of Fit for the Mean. Let $X_1, \ldots, X_n$ be a random sample of size n from an unknown r-variate distribution F_0 having mean μ_0 and nonsingular covariance matrix Σ_0. The jth component of X_i is denoted by X_i^j, $j = 1, \ldots, r$, so that $X_i = (X_i^1, \ldots, X_i^r)^T$. Let $\bar{CR}$ be the empirical goodness of fit for the mean; for a specified vector $\mu = (\mu^1, \ldots, \mu^r)^T$, $\bar{CR}(\mu)$ is defined to be the minimum value of

$$\frac{2}{\lambda(\lambda+1)} \sum ((np^i)^{-\lambda} - 1)$$

over all vectors $p = (p^1, \ldots, p^n)$ such that $\sum p^i = 1$ and $\sum X_i p^i = \mu$.

An explicit expression for $\bar{CR}(\mu)$ can easily be derived using Lagrange multipliers. The extreme value is obtained when

$$p^i = p^i(\mu) = n^{-1}\{1 + s + t^T(X_i - \mu)\}^{-1/(\lambda+1)}$$

where s is a constant given by

$$\sum n^{-1}\{1 + s + t^T(X_i - \mu)\}^{-1/(\lambda+1)} = 1$$

and t is an r-vector given by

$$\sum n^{-1}\{1+s+t^T(X_i-\mu)\}^{-1/(\lambda+1)}(X_i-\mu)=0.$$

We denote the empirical goodness of fit statistic for the mean by $\bar{CR}_0 = \bar{CR}(\mu_0)$.

Let $Y_1,\ldots,Y_n$ be defined by $Y_i=\Sigma_0^{-1/2}(X_i-\mu_0)$, let $u=\Sigma_0^{1/2}t$ and let $\nu=\Sigma_0^{-1/2}(\mu-\mu_0)$. Then

$$\{1+s+t^T(X_i-\mu)\}=\{1+s+u^T(Y_i-\nu)\},$$

and

$$\sum\{1+s+u^T(Y_i-\nu)\}^{-1/(\lambda+1)}(Y_i-\nu)=0,$$

so without loss of generality we may assume that $\mu_0=0$ and $\Sigma_0=I$. This assumption simplifies the algebra in the later steps.

Step 2. Expansion of $\bar{CR}_0$. It is convenient for the expansions that follow to introduce the notation

$$\alpha^{j_1\cdots j_k}=E\left[\{\Sigma_0^{-1/2}(X_i-EX_i)\}^{j_1}\cdots\{\Sigma_0^{-1/2}(X_i-EX_i)\}^{j_k}\right]=E(X_i^{j_1}\cdots X_i^{j_k}),$$

(the standardized moments) and

$$\begin{aligned} A^{j_1\cdots j_k} &= n^{-1}\sum[\{\Sigma_0^{-1/2}(X_i-EX_i)\}^{j_1}\cdots\{\Sigma_0^{-1/2}(X_i-EX_i)\}^{j_k}]-\alpha^{j_1\cdots j_k}\\ &= n^{-1}\sum(X_i^{j_1}\cdots X_i^{j_k}-\alpha^{j_1\cdots j_k}),\end{aligned}$$

the discrepancies between the sample moments and the true moments. As the X_i are standardized, $\alpha^j=0$, $A^j=n^{-1}\sum X_i^j$, and $\alpha^{jk}=\delta^{jk}$, $(j,k=1,\ldots,r)$, where δ^{jk} is Kronecker's delta. Furthermore, $A^{j_1\cdots j_k}$ is $O_p(n^{-1/2})$ (moments of the appropriate order are assumed to exist). Use is made here of the usual convention in which summation over repeated indices is understood. An equivalence which we shall use heavily below is

$$n^{-1}\sum X^{j_1\cdots j_k}=\alpha^{j_1\cdots j_k}+A^{j_1\cdots j_k}.$$

The first stage of this step is to express the constant s in terms of the vector t. Given that s is $O_p(n^{-1})$ and t is $O_p(n^{-1/2})$ (shown elsewhere), we expand

$$\sum n^{-1}\{1+s+t^T(X_i)\}^{-1/(\lambda+1)}=1$$

in a Taylor series to get

$$\begin{aligned}
0 &= \frac{-1}{(\lambda+1)}(s+t^j(A^j+\alpha^j)) + \frac{(\lambda+2)}{2(\lambda+1)^2}(s^2+2st^j(A^j+\alpha^j)+t^jt^k(A^{jk}+\alpha^{jk}))\\
&\quad -\frac{(\lambda+2)(2\lambda+3)}{6(\lambda+1)^3}(s^3+3s^2t^j(A^j+\alpha^j)+3st^jt^k(A^{jk}+\alpha^{jk})+t^jt^kt^l(A^{jkl}+\alpha^{jkl}))\\
&\quad +\frac{(\lambda+2)(2\lambda+3)(3\lambda+4)}{24(\lambda+1)^4}(s^4+4s^3t^j(A^j+\alpha^j)+6s^2t^jt^k(A^{jk}+\alpha^{jk})\\
&\quad +4st^jt^kt^l(A^{jkl}+\alpha^{jkl})+t^jt^kt^lt^m(A^{jklm}+\alpha^{jklm}))+O_p(n^{-5/2})\\
&= \frac{-1}{(\lambda+1)}(s+A^jt^j)+\frac{(\lambda+2)}{2(\lambda+1)^2}(s^2+2sA^jt^j+A^{jk}t^jt^k+t^jt^j)\\
&\quad -\frac{(\lambda+2)(2\lambda+3)}{6(\lambda+1)^3}(3st^jt^j+A^{jkl}t^jt^kt^l+\alpha^{jkl}t^jt^kt^l)\\
&\quad +\frac{(\lambda+2)(2\lambda+3)(3\lambda+4)}{24(\lambda+1)^4}\alpha^{jklm}t^jt^kt^lt^m+O_p(n^{-5/2}).
\end{aligned}$$

Multiplying by $(\lambda+1)$ and pulling the first s to the left-hand side of the equation,

$$\begin{aligned}
s &= -A^jt^j+\frac{(\lambda+2)}{2(\lambda+1)}(s^2+2sA^jt^j+A^{jk}t^jt^k+t^jt^j)\\
&\quad -\frac{(\lambda+2)(2\lambda+3)}{6(\lambda+1)^2}(3st^jt^j+A^{jkl}t^jt^kt^l+\alpha^{jkl}t^jt^kt^l)\\
&\quad +\frac{(\lambda+2)(2\lambda+3)(3\lambda+4)}{24(\lambda+1)^3}\alpha^{jklm}t^jt^kt^lt^m+O_p(n^{-5/2}).
\end{aligned}$$

Expanding all occurrences of s on the right to $O_p(n^{-5/2})$,

$$\begin{aligned}
s^2 &= A^jA^kt^jt^k-2\frac{(\lambda+2)}{2(\lambda+1)}A^jt^jt^kt^k+\frac{(\lambda+2)^2}{4(\lambda+1)^2}t^jt^jt^kt^k++O_p(n^{-5/2}),\\
2sA^jt^j &= -2A^jA^kt^jt^k+2\frac{(\lambda+2)}{2(\lambda+1)}A^jt^jt^kt^k+O_p(n^{-5/2}),\\
3st^jt^j &= -3A^jt^jt^kt^k+\frac{3(\lambda+2)}{2(\lambda+1)}t^jt^jt^kt^k+O_p(n^{-5/2}),
\end{aligned}$$

and

$$\begin{aligned}
s &= -A^jt^j+\frac{(\lambda+2)}{2(\lambda+1)}(-A^jA^kt^jt^k+\frac{(\lambda+2)^2}{4(\lambda+1)^2}t^jt^jt^kt^k+A^{jk}t^jt^k+t^jt^j)\\
&\quad -\frac{(\lambda+2)(2\lambda+3)}{6(\lambda+1)^2}(-3A^jt^jt^kt^k+\frac{3(\lambda+2)}{2(\lambda+1)}t^jt^jt^kt^k+A^{jkl}t^jt^kt^l+\alpha^{jkl}t^jt^kt^l)\\
&\quad +\frac{(\lambda+2)(2\lambda+3)(3\lambda+4)}{24(\lambda+1)^3}\alpha^{jklm}t^jt^kt^lt^m+O_p(n^{-5/2}).
\end{aligned}$$

We now obtain an expansion for t which solves

$$T=n^{-1}\sum\{1+s+t^TX_i\}^{-1/(\lambda+1)}X_i=0$$

to order $O_p(n^{-2})$. Again expanding $(1+s+t^T X_i)^{-1/(\lambda+1)}$ in a Taylor series, we get

$$\begin{aligned}
T^j &= A^j+\alpha^j-\frac{1}{(\lambda+1)}(s(A^j+\alpha^j)+t^k(A^{jk}+\alpha^{jk})) \\
&+\frac{(\lambda+2)}{2(\lambda+1)^2}(s^2(A^j+\alpha^j)+2st^k(A^{jk}+\alpha^{jk})+t^kt^l(A^{jkl}+\alpha^{jkl})) \\
&-\frac{(\lambda+2)(2\lambda+3)}{6(\lambda+1)^3}(s^3(A^j+\alpha^j)+3s^2t^k(A^{jk}+\alpha^{jk}) \\
&+3st^kt^l(A^{jkl}+\alpha^{jkl})+t^kt^lt^m(A^{jklm}+\alpha^{jklm}))+O_p(n^{-2}) \\
&= A^j-\frac{1}{(\lambda+1)}(sA^j+A^{jk}t^k+t^j)+\frac{(\lambda+2)}{2(\lambda+1)^2}(2st^j+A^{jkl}t^kt^l+\alpha^{jkl}t^kt^l) \\
&-\frac{(\lambda+2)(2\lambda+3)}{6(\lambda+1)^3}\alpha^{jklm}t^kt^lt^m+O_p(n^{-2}).
\end{aligned}$$

Plugging in the expression for s found above, this becomes

$$\begin{aligned}
T^j &= A^j-\frac{1}{(\lambda+1)}(-A^jA^kt^k+\frac{(\lambda+2)}{2(\lambda+1)}A^jt^kt^k+A^{jk}t^k+t^j) \\
&+\frac{(\lambda+2)}{2(\lambda+1)^2}(-2A^kt^jt^k+\frac{(\lambda+2)}{(\lambda+1)}t^jt^kt^k+A^{jkl}t^kt^l+\alpha^{jkl}t^kt^l) \\
&-\frac{(\lambda+2)(2\lambda+3)}{6(\lambda+1)^3}\alpha^{jklm}t^kt^lt^m+O_p(n^{-2}).
\end{aligned}$$

Solving $T=0$, we multiply by $(\lambda+1)$ and pull the first t^j over to the left-hand side of the equation to get

$$\begin{aligned}
t^j &= (\lambda+1)A^j-(-A^jA^kt^k+\frac{(\lambda+2)}{2(\lambda+1)}A^jt^kt^k+A^{jk}t^k) \\
&+\frac{(\lambda+2)}{2(\lambda+1)}(-2A^kt^jt^k+\frac{(\lambda+2)}{(\lambda+1)}t^jt^kt^k+A^{jkl}t^kt^l+\alpha^{jkl}t^kt^l) \\
&-\frac{(\lambda+2)(2\lambda+3)}{6(\lambda+1)^2}\alpha^{jklm}t^kt^lt^m+O_p(n^{-2}).
\end{aligned}$$

Expanding occurrences of t^j and t^k in $O_p(n^{-3/2})$ terms, the equation above reduces to

$$\begin{aligned}
t^j &= (\lambda+1)A^j-(-(\lambda+1)A^jA^kA^k+\frac{(\lambda+2)(\lambda+1)}{2}A^jA^kA^k+A^{jk}t^k) \\
&+\frac{(\lambda+2)}{2(\lambda+1)}(-2(\lambda+1)^2A^jA^kA^k+(\lambda+2)(\lambda+1)^2A^jA^kA^k+ \\
&(\lambda+1)^2A^{jkl}A^kA^l+\alpha^{jkl}t^kt^l)-\frac{(\lambda+1)(\lambda+2)(2\lambda+3)}{6}\alpha^{jklm}A^kA^lA^m+O_p(n^{-2}) \\
&= (\lambda+1)A^j-[\frac{\lambda(\lambda+1)}{2}A^jA^kA^k+A^{jk}t^k]+[\frac{\lambda(\lambda+1)(\lambda+2)}{2}A^jA^kA^k \\
&+\frac{(\lambda+1)(\lambda+2)}{2}A^{jkl}A^kA^l+\frac{(\lambda+2)}{2(\lambda+1)}\alpha^{jkl}t^kt^l] \\
&-\frac{(\lambda+1)(\lambda+2)(2\lambda+3)}{6}\alpha^{jklm}A^kA^lA^m+O_p(n^{-2})
\end{aligned}$$

and finally,

$$\begin{aligned} t^j &= (\lambda+1)A^j + \frac{\lambda(\lambda+1)^2}{2}A^jA^kA^k - A^{jk}t^k + \frac{(\lambda+1)(\lambda+2)}{2}A^{jkl}A^kA^l \\ &\quad + \frac{(\lambda+2)}{2(\lambda+1)}\alpha^{jkl}t^kt^l - \frac{(\lambda+1)(\lambda+2)(2\lambda+3)}{6}\alpha^{jklm}A^kA^lA^m + O_p(n^{-2}). \end{aligned}$$

Now,

$$\begin{aligned} A^{jk}t^k &= A^{jk}((\lambda+1)A^k - A^{kl}t^l + \frac{(\lambda+2)}{2(\lambda+1)}\alpha^{klm}t^lt^m) + O_p(n^{-2}) \\ &= (\lambda+1)A^{jk}A^k - (\lambda+1)A^{jk}A^{kl}A^l + \frac{(\lambda+1)(\lambda+2)}{2}A^{jk}\alpha^{klm}A^lA^m + O_p(n^{-2}) \end{aligned}$$

and

$$\begin{aligned} \alpha^{jkl}t^kt^l &= \alpha^{jkl}t^k((\lambda+1)A^l - A^{lm}t^m + \frac{(\lambda+2)}{2(\lambda+1)}\alpha^{lmn}t^mt^n) + O_p(n^{-2}) \\ &= (\lambda+1)\alpha^{jkl}t^kA^l - (\lambda+1)^2\alpha^{jkl}A^kA^{lm}A^m + \frac{(\lambda+1)^2(\lambda+2)}{2}\alpha^{jkl}A^k\alpha^{lmn}A^mA^n \\ &\quad + O_p(n^{-2}) \\ &= (\lambda+1)^2\alpha^{jkl}A^kA^l - (\lambda+1)\alpha^{jkl}A^lA^{km}t^m + \frac{(\lambda+2)}{2}\alpha^{jkl}A^l\alpha^{kmn}t^mt^n \\ &\quad - (\lambda+1)^2\alpha^{jkl}A^kA^{lm}A^m + \frac{(\lambda+1)^2(\lambda+2)}{2}\alpha^{jkl}A^k\alpha^{lmn}A^mA^n + O_p(n^{-2}) \\ &= (\lambda+1)^2\alpha^{jkl}A^kA^l - 2(\lambda+1)^2\alpha^{jkl}A^lA^{km}A^m \\ &\quad + (\lambda+1)^2(\lambda+2)\alpha^{jkl}\alpha^{kmn}A^lA^mA^n + O_p(n^{-2}). \end{aligned}$$

Plugging these into the expression for t^j above,

$$\begin{aligned} t^j &= (\lambda+1)A^j + \frac{\lambda(\lambda+1)^2}{2}A^jA^kA^k - (\lambda+1)A^{jk}A^k + (\lambda+1)A^{jk}A^{kl}A^l \\ &\quad - \frac{(\lambda+1)(\lambda+2)}{2}A^{jk}\alpha^{klm}A^lA^m + \frac{(\lambda+1)(\lambda+2)}{2}A^{jkl}A^kA^l + \frac{(\lambda+1)(\lambda+2)}{2}\alpha^{jkl}A^kA^l \\ &\quad - (\lambda+1)(\lambda+2)\alpha^{jkl}A^lA^{km}A^m + \frac{(\lambda+1)(\lambda+2)^2}{2}\alpha^{jkl}\alpha^{kmn}A^lA^mA^n \\ &\quad - \frac{(\lambda+1)(\lambda+2)(2\lambda+3)}{6}\alpha^{jklm}A^kA^lA^m + O_p(n^{-2}) \\ &= (\lambda+1)\Bigg[A^j + \frac{\lambda(\lambda+1)}{2}A^jA^kA^k - A^{jk}A^k + A^{jk}A^{kl}A^l - \frac{(\lambda+2)}{2}A^{jk}\alpha^{klm}A^lA^m \\ &\quad + \frac{(\lambda+2)}{2}A^{jkl}A^kA^l + \frac{(\lambda+2)}{2}\alpha^{jkl}A^kA^l - (\lambda+2)\alpha^{jkl}A^lA^{km}A^m \\ &\quad + \frac{(\lambda+2)^2}{2}\alpha^{jkl}\alpha^{kmn}A^lA^mA^n - \frac{(\lambda+2)(2\lambda+3)}{6}\alpha^{jklm}A^kA^lA^m\Bigg] + O_p(n^{-2}). \end{aligned}$$

Now, multiplying the constraining equation by the constant vector t, we get

$$\sum\{1+s+t^TX_i\}^{-1/(\lambda+1)}t^TX_i = 0,$$

which implies

$$\begin{aligned}
\sum t^TX_i &= \sum\Big[\frac{1}{(\lambda+1)}(st^TX_i+(t^TX_i)^2)-\frac{(\lambda+2)}{2(\lambda+1)^2}(s^2t^TX_i+2s(t^TX_i)^2+(t^TX_i)^3)\\
&\quad+\frac{(\lambda+2)(2\lambda+3)}{6(\lambda+1)^3}(s^3t^TX^i+3s^2(t^TX_i)^2+3s(t^TX_i)^3+(t^TX_i)^4)\Big]\\
&\quad+O_p(n^{-3/2})\\
&= \frac{1}{(\lambda+1)}\sum\Big[st^TX_i+(t^TX_i)^2-\frac{(\lambda+2)}{(\lambda+1)}s(t^TX_i)^2-\frac{(\lambda+2)}{2(\lambda+1)}(t^TX_i)^3\\
&\quad+\frac{(\lambda+2)(2\lambda+3)}{6(\lambda+1)^2}(t^TX_i)^4\Big]+O_p(n^{-3/2}).
\end{aligned}$$

As

$$\begin{aligned}
\bar{CR}_0 &= \frac{2}{\lambda(\lambda+1)}\sum((1+s+t^TX_i)^{\lambda/(\lambda+1)}-1)\\
&= \frac{2}{\lambda(\lambda+1)}\sum\Big[\frac{\lambda}{(\lambda+1)}(s+t^TX_i)-\frac{\lambda}{2(\lambda+1)^2}(s^2+2st^TX_i+(t^TX_i)^2)\\
&\quad+\frac{\lambda(\lambda+2)}{6(\lambda+1)^3}(s^3+3s^2t^TX_i+3s(t^TX_i)^2+(t^TX_i)^3)\\
&\quad-\frac{\lambda(\lambda+2)(2\lambda+3)}{24(\lambda+1)^4}(s^4+4s^3t^TX_i+6s^2(t^TX_i)^2+4s(t^TX_i)^3+(t^TX_i)^4)\Big]\\
&\quad+O_p(n^{-3/2})\\
&= \frac{2}{(\lambda+1)^2}\sum\Big[(s+t^TX_i)-\frac{1}{2(\lambda+1)}(s^2+2st^TX_i+(t^TX_i)^2)\\
&\quad+\frac{(\lambda+2)}{6(\lambda+1)^2}(3s(t^TX_i)^2+(t^TX_i)^3)\\
&\quad-\frac{(\lambda+2)(2\lambda+3)}{24(\lambda+1)^3}(t^TX_i)^4\Big]+O_p(n^{-3/2})
\end{aligned}$$

it follows (replacing $\sum t^TX_i$ with the expression found above) that

$$\begin{aligned}
n^{-1}\bar{CR}_0 &= \frac{2}{n(\lambda+1)^2}\sum\Big[s+\frac{s}{(\lambda+1)}t^TX_i+\frac{1}{(\lambda+1)}(t^TX_i)^2-\frac{(\lambda+2)}{(\lambda+1)^2}s(t^TX_i)^2\\
&\quad-\frac{(\lambda+2)}{2(\lambda+1)^2}(t^TX_i)^3+\frac{(\lambda+2)(2\lambda+3)}{6(\lambda+1)^3}(t^TX_i)^4-\frac{s^2}{2(\lambda+1)}\\
&\quad-\frac{s}{(\lambda+1)}t^TX_i-\frac{1}{2(\lambda+1)}(t^TX_i)^2+\frac{(\lambda+2)}{2(\lambda+1)^2}s(t^TX_i)^2\\
&\quad+\frac{(\lambda+2)}{6(\lambda+1)^2}(t^TX_i)^3-\frac{(\lambda+2)(2\lambda+3)}{24(\lambda+1)^3}(t^TX_i)^4\Big]+O_p(n^{-5/2})
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2}{n(\lambda+1)^2}\sum\left[s+\frac{1}{2(\lambda+1)}(t^TX_i)^2-\frac{(\lambda+2)}{2(\lambda+1)^2}s(t^TX_i)^2\right.\\
&\quad\left.-\frac{(\lambda+2)}{3(\lambda+1)^2}(t^TX_i)^3+\frac{(\lambda+2)(2\lambda+3)}{8(\lambda+1)^3}(t^TX_i)^4-\frac{s^2}{2(\lambda+1)}\right]+O_p(n^{-5/2})\\
&= \frac{2}{(\lambda+1)^2}\left[s+\frac{1}{2(\lambda+1)}t^jt^k(A^{jk}+\alpha^{jk})-\frac{(\lambda+2)}{2(\lambda+1)^2}st^jt^k(A^{jk}+\alpha^{jk})\right.\\
&\quad-\frac{(\lambda+2)}{3(\lambda+1)^2}t^jt^kt^l(A^{jkl}+\alpha^{jkl})+\frac{(\lambda+2)(2\lambda+3)}{8(\lambda+1)^3}t^jt^kt^lt^m(A^{jklm}+\alpha^{jklm})\\
&\quad\left.-\frac{s^2}{2(\lambda+1)}\right]+O_p(n^{-5/2})
\end{aligned}
$$

which expands (replacing s with the expression found above to the appropriate order) to

$$
\begin{aligned}
n^{-1}\bar{CR}_0 &= \frac{2}{(\lambda+1)^2}\left[-A^jt^j-\frac{(\lambda+2)}{2(\lambda+1)}A^jA^kt^jt^k+\frac{(\lambda+2)^3}{8(\lambda+1)^3}t^jt^jt^kt^k+\frac{(\lambda+2)}{2(\lambda+1)}A^{jk}t^jt^k\right.\\
&\quad+\frac{(\lambda+2)}{2(\lambda+1)}t^jt^j+\frac{(\lambda+2)(2\lambda+3)}{2(\lambda+1)^2}A^kt^kt^jt^j-\frac{(\lambda+2)^2(2\lambda+3)}{4(\lambda+1)^3}t^jt^jt^kt^k\\
&\quad-\frac{(\lambda+2)(2\lambda+3)}{6(\lambda+1)^2}A^{jkl}t^jt^kt^l-\frac{(\lambda+2)(2\lambda+3)}{6(\lambda+1)^2}\alpha^{jkl}t^jt^kt^l\\
&\quad+\frac{(\lambda+2)(2\lambda+3)(3\lambda+4)}{24(\lambda+1)^3}\alpha^{jklm}t^jt^kt^lt^m\\
&\quad+\frac{1}{2(\lambda+1)}A^{jk}t^jt^k+\frac{1}{2(\lambda+1)}\alpha^{jk}t^jt^k-\frac{(\lambda+2)}{2(\lambda+1)^2}A^{jk}st^jt^k-\frac{(\lambda+2)}{2(\lambda+1)^2}\alpha^{jk}st^jt^k\\
&\quad-\frac{(\lambda+2)}{3(\lambda+1)^2}A^{jkl}t^jt^kt^l-\frac{(\lambda+2)}{3(\lambda+1)^2}\alpha^{jkl}t^jt^kt^l\\
&\quad+\frac{(\lambda+2)(2\lambda+3)}{8(\lambda+1)^3}A^{jklm}t^jt^kt^lt^m+\frac{(\lambda+2)(2\lambda+3)}{8(\lambda+1)^3}\alpha^{jklm}t^jt^kt^lt^m\\
&\quad\left.-\frac{1}{2(\lambda+1)}A^jA^kt^jt^k+\frac{(\lambda+2)}{2(\lambda+1)^2}A^jt^jt^kt^k-\frac{(\lambda+2)^2}{8(\lambda+1)^3}t^jt^jt^kt^k\right]+O_p(n^{-5/2}).(1)
\end{aligned}
$$

Taking only the $O_p(n^{-1})$ terms from 1,

$$\frac{2}{(\lambda+1)^2}[-A^jt^j+\frac{(\lambda+2)}{2(\lambda+1)}t^jt^j+\frac{1}{2(\lambda+1)}\alpha^{jk}t^jt^k],$$

and rewriting them as

$$A^jA^j+\left[A^j+\frac{(\lambda+3)}{(\lambda+1)^2}t^j\right]\left[\frac{t^j}{(\lambda+1)}-A^j\right],$$

these expand to

$$
\begin{aligned}
A^jA^j \quad &+ \quad (A^j+\frac{(\lambda+3)}{(\lambda+1)^2}t^j)\left[\frac{\lambda(\lambda+1)}{2}A^jA^kA^k-A^{jk}A^k+A^{jk}A^{kl}A^l\right.\\
&\quad-\frac{(\lambda+2)}{2}A^{jk}\alpha^{klm}A^lA^m+\frac{(\lambda+2)}{2}A^{jkl}A^kA^l+\frac{(\lambda+2)}{2}\alpha^{jkl}A^kA^l
\end{aligned}
$$

$$
\begin{aligned}
&-(\lambda+2)\alpha^{jkl}A^lA^{km}A^m+\frac{(\lambda+2)^2}{2}\alpha^{jkl}\alpha^{kmn}A^lA^mA^n \\
&-\frac{(\lambda+2)(2\lambda+3)}{6}\alpha^{jklm}A^kA^lA^m\Big]+O_p(n^{-5/2})
\end{aligned} \tag{2}
$$

where the A^jA^j term is the only one that is $O(n^{-1})$. Next, taking only the $O(n^{-3/2})$ terms from 1 and 2,

$$
\begin{aligned}
&\frac{(\lambda+3)}{(\lambda+1)^3}A^{jk}t^jt^k-\frac{(\lambda+2)(2\lambda+5)}{3(\lambda+1)^4}\alpha^{jkl}t^jt^kt^l-A^{jk}A^jA^k \\
&+\frac{(\lambda+2)}{2}\alpha^{jkl}A^jA^kA^l-\frac{(\lambda+3)}{(\lambda+1)^2}A^{jk}t^jA^k+\frac{(\lambda+3)(\lambda+2)}{2(\lambda+1)^2}\alpha^{jkl}t^jA^kA^l,
\end{aligned}
$$

these expand to

$$
\begin{aligned}
&-A^{jk}A^jA^k+\frac{(\lambda+2)}{3}\alpha^{jkl}A^jA^kA^l \\
&-\frac{(\lambda+3)}{(\lambda+1)}A^{jk}A^{jl}A^kA^l+\frac{(\lambda+2)^2}{(\lambda+1)}\alpha^{jlm}A^{jk}A^kA^lA^m \\
&-\frac{(\lambda+2)^2(3\lambda+7)}{(\lambda+1)}\alpha^{jkl}\alpha^{kmn}A^jA^lA^mA^n+O_p(n^{-5/2})
\end{aligned} \tag{3}
$$

where only the first two terms are $O_p(n^{-3/2})$. Finally, combining the $O_p(n^{-2})$ terms from 1, 2 and 3 and adding them to the $O_p(n^{-1})$ and $O_p(n^{-3/2})$ terms just obtained yields

$$
\begin{aligned}
n^{-1}\bar{CR}_0 \;=\;& A^jA^j-A^{jk}A^jA^k+\frac{(\lambda+2)}{3}\alpha^{jkl}A^jA^kA^l+\frac{\lambda^2}{4}A^jA^jA^kA^k+A^{jk}A^{kl}A^lA^j \\
&+\frac{(\lambda+2)}{3}A^{jkl}A^jA^kA^l+\frac{(\lambda+2)^2}{4}\alpha^{jkl}\alpha^{kmn}A^jA^lA^mA^n \\
&-\frac{(\lambda+2)(2\lambda+3)}{12}\alpha^{jklm}A^jA^kA^lA^m-(\lambda+2)\alpha^{klm}A^jA^{jk}A^lA^m \\
&+O_p(n^{-5/2}).
\end{aligned} \tag{4}
$$

When $\lambda=0$, this equation reduces to that found by DHR for empirical likelihood (up to a rearrangement of dummy indices).

Step 3. Expansion of $\bar{CR}(\mu)$. It is evident that the preceding expansion for $\bar{CR}_0$ can be used to develop a similar expansion for $\bar{CR}(\mu)$ by replacing X_i with $X_i-\mu$ throughout. Assuming that μ is $O_p(n^{-1/2})$, this process involves replacing A^j with $A^j-\mu^j$, replacing A^{jk} with $A^{jk}-(\mu^jA^k+\mu^kA^j)+\mu^j\mu^k=A^{jk}+O_p(n^{-1})$, and replacing A^{jkl} with $A^{jkl}-(\mu^j\alpha^{kl}+\mu^k\alpha^{jl}+\mu^l\alpha^{jk})+O_p(n^{-1})$. Let $A=\bar{X}=(A^j)$. Then

$$
n^{-1}\bar{CR}_0 \;=\; (A-\mu)^j(A-\mu)^j
$$

$$
\begin{aligned}
&-\{A^{jk}-(\mu^j A^k+\mu^k A^j)+\mu^j\mu^k\}(A-\mu)^j(A-\mu)^k \\
&+\frac{(\lambda+2)}{3}\alpha^{jkl}(A-\mu)^j(A-\mu)^k(A-\mu)^l \\
&+\frac{\lambda^2}{4}(A-\mu)^j(A-\mu)^j(A-\mu)^k(A-\mu)^k \\
&+A^{jk}A^{kl}(A-\mu)^l(A-\mu)^j \\
&+\frac{(\lambda+2)}{3}\{A^{jkl}-(\mu^j\alpha^{kl}+\mu^k\alpha^{jl}+\mu^l\alpha^{jk})\}(A-\mu)^j(A-\mu)^k(A-\mu)^l \\
&+\frac{(\lambda+2)^2}{4}\alpha^{jkl}\alpha^{kmn}(A-\mu)^j(A-\mu)^l(A-\mu)^m(A-\mu)^n \\
&-\frac{(\lambda+2)(2\lambda+3)}{12}\alpha^{jklm}(A-\mu)^j(A-\mu)^k(A-\mu)^l(A-\mu)^m \\
&-(\lambda+2)\alpha^{klm}A^{jk}(A-\mu)^j(A-\mu)^l(A-\mu)^m+O_p(n^{-5/2}). \qquad (5)
\end{aligned}
$$

Again, when $\lambda = 0$, this expression agrees with that of DHR.

Step 4. Empirical Goodness of Fit for a function of the mean. Suppose that $\theta = (\theta^1, \ldots, \theta^q)^T$ is a q-dimensional parameter ($q \leq r$) that depends on the underlying distribution only through its mean, and let $\theta_0 = \theta(\mu_0) = \theta(0)$. The empirical goodness of fit statistic for this parameter, denoted by CR_0, can be obtained as the minimum of $\bar{CR}(\mu)$ over all vectors μ that satisfy $\theta(\mu) = \theta_0$. An expansion of CR_0 can thus be derived from expression 5.

Let $\tilde{\mu}$ be the value of μ that minimizes $\bar{CR}(\mu)$ subject to the constraint $\theta(\mu) = \theta_0$. Consider the expansion $\tilde{\mu} = A - (\mu_1+\mu_2+\mu_3)+O_p(n^{-2})$ where μ_1, μ_2 and μ_3 are $O_p(n^{-1/2})$, $O_p(n^{-1})$ and $O_p(n^{-3/2})$, respectively. We now deternmine μ_1, μ_2 and μ_3 successively. For these derivations, let

$$\theta^u_{j_1\cdots j_k} = \partial^k\theta(\mu)/\partial\mu^{j_1}\ldots\partial\mu^{j_k}|_{\mu=0}, \quad u = 1, \ldots, q,$$

and let $\Theta = (\theta^u_j)$, so that Θ is a $q \times r$ matrix. It is assumed that Θ is of full rank q.

If μ is $O_p(n^{-1/2})$, then $n^{-1}\bar{CR}(\mu) = (A-\mu)^j(A-\mu)^j + O_p(n^{-3/2})$, and the constraint $\theta(\mu) = \theta_0$ is equivalent to $\theta^u_j\mu^j = O_p(n^{-1})$, $u = 1, \ldots, q$. (This follows from truncating the multivariate Taylor series expansion for θ_0 about 0 after the first term: $\theta(\mu)^u = \theta^u_0 + \theta^u_j\mu^j + O_p(n^{-1})$.) Solving for μ_1,

$$\min(A-\mu)^j(A-\mu)^j = (\mu_1)^j(\mu_1)^j,$$

requires minimizing $\mu^j_1\mu^j_1$ subject to $\Theta(A-\mu_1) = 0$. (This constraint follows from $\Theta\mu =$

$\Theta(A-\mu_1)+O_p(n^{-1})$.) A Lagrange multiplier argument,

$$\min \mu_1^T\mu_1+\gamma^T\Theta(A-\mu_1),$$

shows that $\mu_1 = MA$, where $M = \Theta^T(\Theta\Theta^T)^{-1}\Theta$.

Again, if μ is $O_p(n^{-1/2})$, then

$$\begin{aligned} n^{-1}\bar{CR}(\mu) &= (A-\mu)^j(A-\mu)^j - A^{jk}(A-\mu)^j(A-\mu)^k \\ &\quad + \frac{(\lambda+2)}{3}\alpha^{jkl}(A-\mu)^j(A-\mu)^k(A-\mu)^l + O_p(n^{-2}), \end{aligned}$$

and the constraint $\theta(\mu)=\theta_0$ is equivalent to $\theta^u_j\mu^j+\frac{1}{2}\theta^u_{jk}\mu^j\mu^k = O_p(n^{-3/2})$, $u=1,\ldots,q$ (truncating the multivariate Taylor series after two terms). Now, having fixed $\mu_1 = MA$, solving for μ_2 requires minimizing

$$\begin{aligned} &(\mu_1+\mu_2)^j(\mu_1+\mu_2)^j - A^{jk}(\mu_1+\mu_2)^j(\mu_1+\mu_2)^k \\ &+\frac{(\lambda+2)}{3}\alpha^{jkl}(\mu_1+\mu_2)^j(\mu_1+\mu_2)^k(\mu_1+\mu_2)^l, \end{aligned}$$

or, equivalently (dropping $O_p(n^{-5/2})$ terms and those not explicitly involving μ_2)

$$\mu_2^j\mu_2^j + 2\mu_2^j\mu_1^j - 2A^{jk}\mu_2^j\mu_1^k + (\lambda+2)\alpha^{jkl}\mu_2^j\mu_1^k\mu_1^l$$

which we rewrite as

$$\mu_2^j\mu_2^j + 2\mu_2^j(\mu_1^j+B)$$

where $B^j = ((\lambda+2)/2)\alpha^{jkl}\mu_1^k\mu_1^l - A^{jk}\mu_1^k$, subject to $\Theta\mu_2 = U$, where $U^u = \frac{1}{2}\theta^u_{jk}(A-\mu_1)^j(A-\mu_1)^k$. This value for the constraint can be found by noting that

$$\theta(\mu) = \theta_0 + \frac{\partial\theta(\mu)}{\partial\mu^j}|_{\mu=0}\mu^j + \frac{1}{2}\frac{\partial^2\theta(\mu)}{\partial\mu^j\partial\mu^k}|_{\mu=0}\mu^j\mu^k + O_p(n^{-3/2}),$$

and expanding the expressions for μ^j to the appropriate order,

$$\theta(\mu) = \theta_0 + \frac{\partial\theta(\mu)}{\partial\mu^j}|_{\mu=0}(A-\mu_1-\mu_2)^j + \frac{1}{2}\frac{\partial^2\theta(\mu)}{\partial\mu^j\partial\mu^k}|_{\mu=0}(A-\mu_1)^j(A-\mu_1)^k + O_p(n^{-3/2}).$$

As $\Theta A = \Theta\mu_1$, the constraint follows. A Lagrange multiplier argument shows that $\mu_2 = NU-(I-M)B$, where $N = \Theta^T(\Theta\Theta^T)^{-1}$.

Finally, if $\mu = O_p(n^{-1/2})$, then expansion 5 for $n^{-1}\bar{CR}(\mu)$ holds with error of order $O_p(n^{-5/2})$, and the constraint $\theta(\mu)=\theta_0$ is equivalent to $\theta^u_j\mu^j+\frac{1}{2}\theta^u_{jk}\mu^j\mu^k+\frac{1}{6}\theta^u_{jkl}\mu^j\mu^k\mu^l =$

$O_p(n^{-2})$, $u = 1, \ldots, q$. Now having fixed $\mu_1 = MA$ and $\mu_2 = NU - (I - M)B$, solving for μ_3 requires minimizing

$$
\begin{aligned}
&\mu_3^j\mu_3^j + 2\mu_3^j\mu_1^j + 2\mu_3^j\mu_2^j \\
&-2A^{jk}\mu_3^j\mu_1^k - 2A^{jk}\mu_3^j\mu_2^k \\
&+(\lambda+2)\alpha^{jkl}\mu_3^j\mu_1^k\mu_1^l + 2(\lambda+2)\alpha^{jkl}\mu_3^j\mu_1^k\mu_2^l \\
&+\lambda^2\mu_3^j\mu_1^j\mu_1^k\mu_1^k \\
&+2A^{jl}A^{kl}\mu_3^j\mu_1^k \\
&+(\lambda+2)A^{jkl}\mu_3^j\mu_1^k\mu_1^l \\
&+(\lambda+2)^2\alpha^{jkn}\alpha^{lmn}\mu_3^j\mu_1^k\mu_1^l\mu_1^m \\
&-\frac{(\lambda+2)(2\lambda+3)}{3}\alpha^{jklm}\mu_3^j\mu_1^k\mu_1^l\mu_1^m \\
&-(\lambda+2)A^{jm}\alpha^{klm}\mu_3^j\mu_1^k\mu_1^l - 2(\lambda+2)\alpha^{jkm}A^{lm}\mu_3^j\mu_1^k\mu_1^l \\
&+(\mu^jA^k + \mu^kA^j)(A-\mu)^j(A-\mu)^k - \mu^j\mu^k(A-\mu)^j(A-\mu)^k \\
&-\frac{(\lambda+2)}{3}(\mu^j\alpha^{kl} + \mu^k\alpha^{jl} + \mu^l\alpha^{jk})(A-\mu)^j(A-\mu)^k(A-\mu)^l.
\end{aligned}
$$

The last two lines in the equation above expand to

$$
\begin{aligned}
&2A^jA^k\mu_3^j\mu_1^k - 4\mu_3^j\mu_1^j\mu_1^k\mu_1^k \\
&-(\lambda+2)A^j\mu_3^j\mu_1^k\mu_1^k - 2(\lambda+2)A^k\mu_3^j\mu_1^j\mu_1^k + 4(\lambda+2)\mu_3^j\mu_1^j\mu_1^k\mu_1^k.
\end{aligned}
$$

These two lines are omitted in DHR. Note, though, that when $\lambda = 0$, these two lines reduce to

$$2\mu_3^j\mu_1^k(A - 2\mu_1)^j(A - \mu_1)^k,$$

which, if $(A - \mu_1)^k$ can be assumed to be $O_p(n^{-1})$, is $O_p(n^{-7/2})$. As it is not obvious that this assumption can be made, we retain the terms here. As it turns out, they will cancel at a later stage.

Now, it may be wondered whether the equation to be minimized for μ_3 is correct, in that the equation involves terms of order $O_p(n^{-3})$, whereas the expression for $\bar{CR}(\mu)$ ignores terms of $O_p(n^{-5/2})$. To see that the expansion does indeed hold, consider: Any $O_p(n^{-5/2})$ term in 5 involves μ_3 only through $\tilde{\mu}$. Thus, the term involves $(\mu_1 + \mu_2 + \mu_3)$, which is $O_p(n^{-1/2})$ due to the μ_1 term. To include an occurrence of μ_3, this order drops to

$O_p(n^{-3/2})$, so that the contribution from an ignored $O_p(n^{-5/2})$ term is at most $O_p(n^{-7/2})$. Hence, the expansion is valid to the order specified.

We rewrite the function to be minimized as

$$\mu_3^j\mu_3^j + 2\mu_3^j(\mu_1^j + \mu_2^j + B^j + C^j)$$

subject to the constraint $\Theta\mu_3 = V$, where the r-vector C is given by

$$\begin{aligned}
C^j &= -A^{jk}\mu_2^k \\
&\quad +(\lambda+2)\alpha^{jkl}\mu_1^k\mu_2^l \\
&\quad +\frac{\lambda^2}{2}\mu_1^j\mu_1^k\mu_1^k \\
&\quad +A^{jl}A^{kl}\mu_1^k \\
&\quad +\frac{(\lambda+2)}{2}A^{jkl}\mu_1^k\mu_1^l \\
&\quad +\frac{(\lambda+2)^2}{2}\alpha^{jkn}\alpha^{lmn}\mu_1^k\mu_1^l\mu_1^m \\
&\quad -\frac{(\lambda+2)(2\lambda+3)}{6}\alpha^{jklm}\mu_1^k\mu_1^l\mu_1^m \\
&\quad -\frac{(\lambda+2)}{2}A^{jm}\alpha^{klm}\mu_1^k\mu_1^l - (\lambda+2)\alpha^{jkm}A^{lm}\mu_1^k\mu_1^l \\
&\quad +A^jA^k\mu_1^k - 2\mu_1^j\mu_1^k\mu_1^k \\
&\quad -\frac{(\lambda+2)}{2}A^j\mu_1^k\mu_1^k - (\lambda+2)A^k\mu_1^k\mu_1^j + 2(\lambda+2)\mu_1^j\mu_1^k\mu_1^k
\end{aligned}$$

and the q-vector V is given by

$$V^u = -\theta^u_{jk}\mu_2^j(A-\mu_1)^k + \frac{1}{6}\theta^u_{jkl}(A-\mu_1)^j(A-\mu_1)^k(A-\mu_1)^l.$$

A Lagrange multiplier argument shows that $\mu_3 = NV - (I-M)C$.

Summarizing the above calculations,

$$\tilde{\mu} = (I-M)(A+B+C) - N(U+V) + O_p(n^{-2}),$$

and

$$\begin{aligned}
n^{-1}\bar{C}R_0 &= n^{-1}\bar{C}R(\tilde{\mu}) \\
&= \mu_1^j\mu_1^j + \mu_2^j\mu_2^j + 2\mu_1^j\mu_2^j + 2\mu_1^j\mu_3^j \\
&\quad -A^{jk}\mu_1^j\mu_1^k - 2A^{jk}\mu_1^j\mu_2^k
\end{aligned}$$

$$
\begin{aligned}
&+\frac{(\lambda+2)}{3}\alpha^{jkl}\mu_1^j\mu_1^k\mu_1^l + (\lambda+2)\alpha^{jkl}\mu_2^j\mu_1^k\mu_1^l \\
&+\frac{\lambda^2}{4}\mu_1^j\mu_1^j\mu_1^k\mu_1^k \\
&+A^{jl}A^{kl}\mu_1^j\mu_1^k \\
&+\frac{(\lambda+2)}{3}A^{jkl}\mu_1^j\mu_1^k\mu_1^l \\
&+\frac{(\lambda+2)^2}{4}\alpha^{jkn}\alpha^{lmn}\mu_1^j\mu_1^k\mu_1^l\mu_1^m \\
&-\frac{(\lambda+2)(2\lambda+3)}{12}\alpha^{jklm}\mu_1^j\mu_1^k\mu_1^l\mu_1^m \\
&-(\lambda+2)\alpha^{jkm}A^{lm}\mu_1^j\mu_1^k\mu_1^l \\
&+A^jA^k\mu_1^j\mu_1^k - \mu_1^j\mu_1^j\mu_1^k\mu_1^k \\
&-(\lambda+2)A^j\mu_1^j\mu_1^k\mu_1^k + (\lambda+2)\mu_1^j\mu_1^j\mu_1^k\mu_1^k + O_p(n^{-5/2}),
\end{aligned}
$$

where again, the last two lines of the above equation are absent in the expansion of DHR.

Step 5. Signed root of $n^{-1}CR_0$. The signed root of $n^{-1}CR_0$ is a q-dimensional vector $R_0 = (R_0^1, \ldots, R_0^q)^T$ such that $R_0^T R_0 = n^{-1}CR_0$. Consider the expansion $R_0 = R_1 + R_2 + R_3 + O_p(n^{-2})$, where R_1, R_2 and R_3 are $O_p(n^{-1/2})$, $O_p(n^{-1})$, and $O_p(n^{-3/2})$, respectively. Let $O = \Theta^T(\Theta\Theta^T)^{-1/2}$, so that O is an $r \times q$ matrix having the properties $OO^T = M$ and $O^T O = I$. Furthermore, let $R = R_1 + R_2 + R_3$.

To order $O(n^{-1})$, $R^T R = R_1^T R_1 + O_p(n^{-3/2})$, and $n^{-1}CR_0 = \mu_1^j\mu_1^j + O_p(n^{-3/2})$. It follows that a valid solution (up to multiplication by rotation matrix) is $R_1 = O^T\mu_1 = O^T A$, and conversely that $\mu_1 = OR_1$. Thus,

$$\mu_1^j = O^{ju}R_1^u = O^{ju}O^{ku}A^k = M^{jk}A^j$$

and

$$R_1^u = O^{ju}\mu_1^j.$$

To order $O(n^{-3/2})$, $R^T R = R_1^T R_1 + 2R_1^T R_2 + O_p(n^{-2})$, and

$$n^{-1}CR_0 = \mu_1^j\mu_1^j + 2\mu_1^j\mu_2^j - A^{jk}\mu_1^j\mu_1^k + \frac{(\lambda+2)}{3}\alpha^{jkl}\mu_1^j\mu_1^k\mu_1^l + O_p(n^{-2}).$$

It follows, using the expression for μ_1 found above, that

$$R_1^u R_2^u = O^{ju}R_1^u\mu_2^j - \frac{1}{2}A^{jk}O^{ju}O^{kv}R_1^u R_1^v + \frac{(\lambda+2)}{6}\alpha^{jkl}O^{ju}O^{kv}O^{lw}R_1^u R_1^v R_1^w.$$

Now, $\mu_2 = NU - (I-M)B$. Let $D = NU$. As $O^T I = O^T = O^T M$, $O^{ju}(I-M)^{jk}$, the jk^{th} element of $O^T(I-M)$, is null. Thus, $O^{ju}\mu_2^j$ reduces to $O^{ju}D^j$, and

$$R_1^u R_2^u = O^{ju} R_1^u D^j - \frac{1}{2} A^{jk} O^{ju} O^{kv} R_1^u R_1^v + \frac{(\lambda+2)}{6} \alpha^{jkl} O^{ju} O^{kv} O^{lw} R_1^u R_1^v R_1^w. \tag{6}$$

Recalling that $U^v = \frac{1}{2}\theta_{kl}^v (A-\mu_1)^k (A-\mu_1)^l$, we expand the above equation and cancel R_1^u. This cancellation is a somewhat questionable step, in that if the various R_1^u, R_1^v, *etc.* pieces do not appear in a symmetric fashion in a given term, then the signed root is not unique. In this case, all such appearances are symmetric, so the issue does not arise, but the point will be revisited in the derivation of R_3^u. There, the root found will not be uniquely defined, but it will turn out not to matter with respect to the main results of this paper. Proceeding with the substitution, we get

$$\begin{aligned} R_2^u &= \frac{1}{2} O^{ju} N^{jv} \theta_{kl}^v (A-\mu_1)^k (A-\mu_1)^l \\ &\quad -\frac{1}{2} A^{jk} O^{ju} M^{kl} A^l + \frac{(\lambda+2)}{6} \alpha^{jkl} O^{ju} M^{km} M^{ln} A^m A^n. \end{aligned}$$

Let $P = (\Theta\Theta^T)^{-1/2}$. Then $O^T N = P$, and

$$\begin{aligned} R_2^u &= \frac{1}{2} P^{uv} \theta_{jk}^v (I-M)^{jl} (I-M)^{km} A^l A^m \\ &\quad -\frac{1}{2} O^{ju} M^{kl} A^{jk} A^l + \frac{(\lambda+2)}{6} \alpha^{jkl} O^{ju} M^{km} M^{ln} A^m A^n. \end{aligned}$$

Finally, to order $O(n^{-2})$, $R^T R = R_1^T R_1 + 2R_1^T R_2 + R_2^T R_2 + 2R_1^T R_3 + O_p(n^{-5/2})$, and $n^{-1} C R_0$ is given by the expansion at the end of step 4. Thus, filtering out the terms already dealt with,

$$\begin{aligned} 2R_1^u R_3^u + R_2^u R_2^u &= \mu_2^j \mu_2^j + 2\mu_1^j \mu_3^j - 2A^{jk} \mu_1^j \mu_2^k + (\lambda+2)\alpha^{jkl} \mu_1^j \mu_1^k \mu_2^l \\ &\quad + \frac{\lambda^2}{4} \mu_1^j \mu_1^j \mu_1^k \mu_1^k + A^{jl} A^{kl} \mu_1^j \mu_1^k + \frac{(\lambda+2)}{3} A^{jkl} \mu_1^j \mu_1^k \mu_1^l \\ &\quad + \frac{(\lambda+2)^2}{4} \alpha^{jkn} \alpha^{lmn} \mu_1^j \mu_1^k \mu_1^l \mu_1^m \\ &\quad - \frac{(\lambda+2)(2\lambda+3)}{12} \alpha^{jklm} \mu_1^j \mu_1^k \mu_1^l \mu_1^m \\ &\quad - (\lambda+2) A^{jm} \alpha^{klm} \mu_1^j \mu_1^k \mu_1^l \\ &\quad + A^j A^k \mu_1^j \mu_1^k - (\lambda+2) A^j \mu_1^j \mu_1^k \mu_1^k + (\lambda+1) \mu_1^j \mu_1^j \mu_1^k \mu_1^k. \end{aligned}$$

We now expand the $\mu_2^j \mu_2^j$ and $R_2^u R_2^u$ terms. Let $D = NU$. As $\mu_2 = D - (I-M)B$,

$$\mu_2^j \mu_2^j = (D^j - (I-M)^{jk} B^k)(D^j - (I-M)^{jl} B^l)$$

$$= \quad D^jD^j - 2D^j(I-M)^{jk}B^k + (I-M)^{jk}(I-M)^{jl}B^kB^l.$$

As $IN = N = MN$, $I^{jk}D^j = M^{jk}D^j = D^k$ and the second term above vanishes. Further, as $(I-M)$ is idempotent, the above expression becomes

$$\begin{aligned}
\mu_2^j\mu_2^j &= D^jD^j + (I-M)^{jk}B^jB^k \\
&= D^jD^j + (I-M)^{jk}[\frac{(\lambda+2)^2}{4}\alpha^{jlm}\alpha^{kno}\mu_1^l\mu_1^m\mu_1^n\mu_1^o - (\lambda+2)\alpha^{jlm}A^{kn}\mu_1^l\mu_1^m\mu_1^n \\
&\quad +A^{jl}A^{km}\mu_1^l\mu_1^m)].
\end{aligned}$$

We now turn to $R_2^TR_2$.

$$\begin{aligned}
R_2^uR_2^u &= \frac{1}{4}A^{jk}A^{lm}O^{ju}O^{lu}O^{kv}O^{mx}R_1^vR_1^x \\
&\quad +\frac{(\lambda+2)^2}{36}\alpha^{jkl}\alpha^{mno}O^{ju}O^{mu}O^{kv}O^{lw}O^{nx}O^{oy!}R_1^vR_1^wR_1^xR_1^y \\
&\quad +O^{ju}O^{ku}D^jD^k - \frac{(\lambda+2)}{6}A^{jk}\alpha^{lmn}O^{ju}O^{lu}O^{kv}O^{mw}O^{nx}R_1^vR_1^wR_1^x \\
&\quad -A^{jk}D^lO^{ju}O^{lu}O^{kv}R_1^v + \frac{(\lambda+2)}{3}\alpha^{jkl}D^mO^{ju}O^{mu}O^{kv}O^{lw}R_1^vR_1^w \\
&= \frac{1}{4}A^{jk}A^{lm}M^{jl}O^{kv}O^{mx}R_1^vR_1^x \\
&\quad +\frac{(\lambda+2)^2}{36}\alpha^{jkl}\alpha^{mno}M^{jm}O^{kv}O^{lw}O^{nx}O^{oy!}R_1^vR_1^wR_1^xR_1^y \\
&\quad +D^jD^j - \frac{(\lambda+2)}{6}A^{jk}\alpha^{lmn}M^{jl}O^{kv}O^{mw}O^{nx}R_1^vR_1^wR_1^x \\
&\quad -A^{jk}D^jO^{kv}R_1^v + \frac{(\lambda+2)}{3}\alpha^{jkl}D^jO^{kv}O^{lw}R_1^vR_1^w
\end{aligned}$$

which, upon rearrangement of dummy indices, becomes

$$\begin{aligned}
R_2^uR_2^u &= \frac{1}{4}A^{jl}A^{km}M^{lm}O^{ju}O^{kv}R_1^uR_1^v \\
&\quad +\frac{(\lambda+2)^2}{36}\alpha^{jkn}\alpha^{lmo}M^{no}O^{ju}O^{kv}O^{lw}O^{mx}R_1^uR_1^vR_1^wR_1^x \\
&\quad +D^jD^j - \frac{(\lambda+2)}{6}A^{jm}\alpha^{kln}M^{mn}O^{ju}O^{kv}O^{lw}R_1^uR_1^vR_1^w \\
&\quad -A^{jk}D^kO^{ju}R_1^u + \frac{(\lambda+2)}{3}\alpha^{jkl}D^lO^{ju}O^{kv}R_1^uR_1^v.
\end{aligned}$$

Plugging these expressions in above and expanding the μ_3 and μ_2 terms, we get

$$\begin{aligned}
2R_1^uR_3^u &= -\frac{1}{4}A^{jl}A^{km}M^{lm}O^{ju}O^{kv}R_1^uR_1^v \\
&\quad -\frac{(\lambda+2)^2}{36}\alpha^{jkn}\alpha^{lmo}M^{no}O^{ju}O^{kv}O^{lw}O^{mx}R_1^uR_1^vR_1^wR_1^x
\end{aligned}$$

$$
\begin{aligned}
&-D^jD^j \\
&+\frac{\lambda+2}{6}A^{jm}\alpha^{kln}M^{mn}O^{ju}O^{kv}O^{lw}R_1^uR_1^vR_1^w \\
&+\frac{1}{2}A^{jk}N^{kv}\theta^v_{lm}(I-M)^{ln}(I-M)^{mo}A^nA^oO^{ju}R_1^u \\
&-\frac{\lambda+2}{6}\alpha^{jkl}N^{lw}\theta^w_{mn}(I-M)^{mo}(I-M)^{np}A^oA^pO^{ju}O^{kv}R_1^uR_1^v \\
&+D^jD^j \\
&+\frac{(\lambda+2)^2}{4}\alpha^{jkn}\alpha^{lmo}(I-M)^{no}O^{ju}O^{kv}O^{lw}O^{mx}R_1^uR_1^vR_1^wR_1^x \\
&-(\lambda+2)A^{jm}\alpha^{kln}(I-M)^{mn}O^{ju}O^{kv}O^{lw}R_1^uR_1^vR_1^w \\
&+A^{jl}A^{km}(I-M)^{lm}O^{ju}O^{kv}R_1^uR_1^v \\
&+\frac{1}{3}N^{jv}\theta^v_{klm}(I-M)^{kn}(I-M)^{lo}(I-M)^{mp}A^nA^oA^pO^{ju}R_1^u \\
&-N^{jv}\theta^v_{kl}N^{kw}\theta^w_{mn}(I-M)^{lo}(I-M)^{mp}(I-M)^{nq}A^oA^pA^qO^{ju}R_1^u \\
&+(\lambda+2)N^{jv}\theta^v_{kl}(I-M)^{km}\alpha^{mno}(I-M)^{lp}A^pO^{ju}O^{nw}O^{ox}R_1^uR_1^vR_1^x \\
&-2N^{jv}\theta^v_{kl}(I-M)^{km}(I-M)^{lo}A^{mn}A^oO^{ju}O^{nw}R_1^uR_1^w \\
&-A^{jk}N^{kv}\theta^v_{lm}(I-M)^{ln}(I-M)^{mo}A^nA^oO^{ju}R_1^u \\
&+(\lambda+2)A^{jm}\alpha^{kln}(I-M)^{mn}O^{ju}O^{kv}O^{lw}R_1^uR_1^vR_1^w \\
&-2A^{jl}A^{km}(I-M)^{lm}O^{ju}O^{kv}R_1^uR_1^v \\
&+\frac{\lambda+2}{2}\alpha^{jkl}N^{lw}\theta^w_{mn}(I-M)^{mo}(I-M)^{np}A^oA^pO^{ju}O^{kv}R_1^uR_1^v \\
&-\frac{(\lambda+2)^2}{2}\alpha^{jkn}\alpha^{lmo}(I-M)^{no}O^{ju}O^{kv}O^{lw}O^{mx}R_1^uR_1^vR_1^wR_1^x \\
&+(\lambda+2)A^{jm}\alpha^{kln}(I-M)^{mn}O^{ju}O^{kv}O^{lw}R_1^uR_1^vR_1^w \\
&+A^{jl}A^{kl}O^{ju}O^{kv}R_1^uR_1^v \\
&+\frac{\lambda+2}{3}A^{jkl}O^{ju}O^{kv}O^{lw}R_1^uR_1^vR_1^w \\
&+\frac{(\lambda+2)^2}{4}\alpha^{jkn}\alpha^{lmn}O^{ju}O^{kv}O^{lw}O^{mx}R_1^uR_1^vR_1^wR_1^x \\
&-\frac{(\lambda+2)(2\lambda+3)}{12}\alpha^{jklm}O^{ju}O^{kv}O^{lw}O^{mx}R_1^uR_1^vR_1^wR_1^x \\
&-(\lambda+2)A^{jm}\alpha^{jkm}O^{ju}O^{kv}O^{lw}R_1^uR_1^vR_1^w \\
&+(\frac{\lambda^2}{4}+\lambda)O^{ju}O^{jv}O^{kw}O^{kx}R_1^uR_1^vR_1^wR_1^x \\
&-\lambda A^jO^{ju}O^{kv}O^{kw}R_1^uR_1^vR_1^w \\
&+(A^jA^k\mu_1^j\mu_1^k-2A^j\mu_1^j\mu_1^k\mu_1^k+\mu_1^j\mu_1^j\mu_1^k\mu_1^k),
\end{aligned}
$$

where the first 6 lines correspond to $-R_2^u R_2^u$, lines 7-10 to $\mu_2^j \mu_2^j$, lines 11-14 to $2\mu_1^j \mu_3^j$, lines 15-17 to $-2A^{jk}\mu_1^j \mu_2^k$, lines 18-20 to $(\lambda+2)\alpha^{jkl}\mu_1^j \mu_1^k \mu_2^l$, lines 21-25 are in one-to-one correspondence with the entries given in the expansion for $2R_1^u R_3^u + R_2^u R_2^u$, and the last three lines correspond to the terms omitted in the expansion of DHR, where only the last line is not obviously zero when $\lambda = 0$. However, this last line can be rewritten as

$$(A-\mu_1)^j (A-\mu_1)^k \mu_1^j \mu_1^k = (I-M)^{jl}(I-M)^{km} A^l A^m O^{ju} O^{kv} R_1^u R_1^v,$$

which reduces to zero because $(I-M)^{jk}O^{ju}$ is null. We now set about reducing the above equation to a more practical form. Lines 3 and 7 cancel. Lines 1, 10, 17 and 21 become

$$(-(I-M)^{lm} + I^{lm} - \frac{1}{4}M^{lm})A^{jl}A^{km}O^{ju}O^{kv}R_1^u R_1^v$$

which collapses to

$$\frac{3}{4}A^{jl}A^{km}M^{lm}O^{ju}O^{kv}R_1^u R_1^v.$$

Lines 2, 8, 19 and 23 become

$$(-\frac{1}{4}(I-M)^{no} + \frac{1}{4}I^{no} - \frac{1}{36}M^{no})(\lambda+2)^2\alpha^{jkn}\alpha^{lmo}O^{ju}O^{kv}O^{lw}O^{mx}$$

which collapses to

$$\frac{2}{9}(\lambda+2)^2\alpha^{jkn}\alpha^{lmo}M^{no}O^{ju}O^{kv}O^{lw}O^{mx}.$$

Lines 4, 9, 16, 20 and 25 become

$$(\frac{\lambda+2}{6}M^{mn} + (\lambda+2)(I-M)^{mn} - (\lambda+2)I^{mn})A^{jm}\alpha^{kln}O^{ju}O^{kv}O^{lw}R_1^u R_1^v R_1^w$$

which collapses to

$$-\frac{5}{6}(\lambda+2)M^{mn}A^{jm}\alpha^{kln}O^{ju}O^{kv}O^{lw}R_1^u R_1^v R_1^w.$$

Lines 5 and 15 reduce to

$$-\frac{1}{2}A^{jk}N^{kv}\theta_{ln}^v(I-M)^{mo}A^n A^o O^{ju} R_1^u$$

and lines 6 and 18 reduce to

$$\frac{\lambda+2}{3}\alpha^{jkl}N^{lw}\theta_{mn}^w(I-M)^{mo}(I-M)^{np}A^o A^p O^{ju}O^{kv}R_1^u R_1^v.$$

All of the other lines are unique and do not reduce further.

Thus, reordering the terms (to bring them into synch with DHR),

$$
\begin{aligned}
R_3^u = & -\frac{1}{4} O^{ju} N^{kv} \theta_{lm}^v (I-M)^{ln} (I-M)^{mo} A^{jk} A^n A^o \\
& + \frac{3}{8} O^{ju} M^{lm} M^{kn} A^{jl} A^{km} A^n \\
& + \frac{\lambda+2}{6} O^{ju} \alpha^{jkl} M^{km} N^{lv} \theta_{no}^v (I-M)^{np} (I-M)^{oq} A^m A^p A^q \\
& + \frac{\lambda+2}{6} O^{ju} M^{km} M^{ln} A^{jkl} A^m A^n \\
& - c_1 \frac{5}{12} (\lambda+2) O^{ju} \alpha^{jkm} M^{mn} M^{ko} M^{lp} A^{ln} A^o A^p \\
& - (1-c_1) \frac{5}{12} (\lambda+2) O^{ju} \alpha^{kln} M^{mn} M^{ko} M^{lp} A^{jm} A^o A^p \\
& + \frac{(\lambda+2)^2}{9} O^{ju} \alpha^{jkn} \alpha^{lmo} M^{no} M^{kp} M^{lq} M^{mr} A^p A^q A^r \\
& - \frac{(\lambda+2)(2\lambda+3)}{24} O^{ju} \alpha^{jklm} M^{kn} M^{lo} M^{mp} A^n A^o A^p \\
& - \frac{1}{2} P^{uv} \theta_{jk}^v N^{jw} \theta_{mn}^w (I-M)^{kl} (I-M)^{mo} (I-M)^{np} A^l A^o A^p \\
& - c_2 P^{uv} \theta_{jk}^v (I-M)^{jl} (I-M)^{km} M^{no} A^l A^{mn} A^o \\
& - (1-c_2) O^{ju} N^{nv} \theta_{kl}^v (I-M)^{km} (I-M)^{lo} M^{np} A^{jm} A^o A^p \\
& + c_3 \frac{(\lambda+2)}{2} P^{uv} \theta_{jk}^v (I-M)^{jl} (I-M)^{km} \alpha^{mno} M^{np} M^{oq} A^l A^p A^q \\
& + (1-c_3) \frac{(\lambda+2)}{2} O^{ju} N^{nv} \theta_{kl}^v (I-M)^{km} (I-M)^{lp} \alpha^{jmo} M^{nq} M^{or} A^p A^q A^r \\
& + \frac{1}{6} P^{uv} \theta_{jkl}^v (I-M)^{jm} (I-M)^{kn} (I-M)^{lo} A^m A^n A^o \\
& + \frac{\lambda^2+4\lambda}{8} O^{ju} M^{jl} M^{km} M^{kn} A^l A^m A^n \\
& - c_4 \frac{\lambda}{2} O^{ju} M^{km} M^{kn} A^j A^m A^n \\
& - (1-c_4) \frac{\lambda}{2} O^{kv} M^{jl} M^{km} A^j A^l A^m
\end{aligned}
$$

where the constants c_1 thru c_4 represent places where a factor of R_1 was cancelled, but the R_1's did not appear in the full equation in a symmetric manner with respect to the other components. Any values of these constants are equally valid. Lines 1-13 reduce to the expression given in DHR (with $c_1 = 1/2$, $c_2 = c_3 = 1$).

Step 6. Formulae for moments. This step contains some formulae that are required for calculating the joint cumulants of R.

Suppose that X is a random vector having the same distribution as X_i, and suppose that $h^1, h^2, \ldots$ are real-valued functions such that $E\{h^j(X)\} = 0, j = 1, 2, \ldots$. Let $H^j = h^j(X)$

and $Z^j = n^{-1}\sum h^j(X_i)$. Then

$$\begin{aligned}
E(Z^jZ^k) &= n^{-1}E(H^jH^k), \\
E(Z^jZ^kZ^l) &= n^{-2}E(H^jH^kH^l), \\
E(Z^jZ^kZ^lZ^m) &= (n^{-2}-n^{-3})E(H^jH^k)E(H^lH^m)[3] + n^{-3}E(H^jH^kH^lH^m), \\
E(Z^jZ^kZ^lZ^mZ^n) &= n^{-3}E(H^jH^k)E(H^lH^mH^n)[10] + O_p(n^{-4}), \\
E(Z^jZ^kZ^lZ^mZ^nZ^o) &= n^{-3}E(H^jH^k)E(H^lH^m)E(H^nH^o)[15] + O_p(n^{-4}),
\end{aligned}$$

where

$$\begin{aligned}
E(H^jH^k)E(H^lH^m)[3] &= E(H^jH^k)E(H^lH^m) + E(H^jH^l)E(H^kH^m) \\
&\quad + E(H^jH^m)E(H^kH^l),
\end{aligned}$$

and similarly for $E(H^jH^k)E(H^lH^mH^n)[10]$ and $E(H^jH^k)E(H^lH^m)E(H^nH^o)[15]$. To see this, consider $E(Z^jZ^k) = n^{-2}E(\sum\sum h^j(X_{i1})h^k(X_{i2}))$. As the X_i are independent, this factors in all cases where $i1 \neq i2$, and these factored elements vanish as the first moment of $h^j(X)$ is zero. Thus, this reduces to $n^{-2}E(\sum h^j(X_i)h^k(X_i)) = n^{-1}E(H^jH^k)$. Similar matching arguments yield the other equalities above.

Step 7. Joint third-order cumulants of R. The joint third order cumulants of R are given (McCullagh, 1987, p.31) by

$$cum(R^u, R^v, R^w) = E(R^uR^vR^w) - E(R^u)E(R^vR^w)[3] + 2E(R^u)E(R^v)E(R^w).$$

Note that R_1 contains one A term, each component of R_2 contains two A terms, and each component of R_3 contains three A terms. By the matching argument given above, all components having five or more A terms have expectations which are $O(n^{-3})$ since the first moment vanishes. Thus, noting that $E(R_1^u) = 0$ (see below), the expression for the joint third-order cumulant reduces to

$$cum(R^u, R^v, R^w) = E(R_1^uR_1^vR_1^w) + E(R_2^uR_1^vR_1^w)[3] - E(R_2^u)E(R_1^vR_1^w)[3] + O(n^{-3}).$$

It follows from the formulae of Steps 5 and 6 that

$$E(R_1^u) \quad = \quad E(A^jO^{ju}) = \alpha^jO^{ju} = 0$$

$$
\begin{aligned}
E(R_1^u R_1^v) &= E(A^j A^k O^{ju} O^{kv}) = n^{-1}\alpha^{jk} O^{ju} O^{kv} = n^{-1} O^{ju} O^{jv} = n^{-1}\delta^{uv} \\
E(R_1^u R_1^v R_1^w) &= E(A^j A^k A^l O^{ju} O^{kv} O^{lw}) = n^{-2}\alpha^{jkl} O^{ju} O^{kv} O^{lw} \\
E(R_2^u) &= n^{-1}\left\{\frac{\lambda-1}{6}\alpha^{jkl} O^{ju} M^{kl} + \frac{1}{2} P^{uv}\theta^v_{jk}(I-M)^{jk}\right\} \\
E(R_2^u R_1^v R_1^w) &= n^{-2}\left\{\frac{1}{2} P^{ux}\theta^x_{jk}(I-M)^{jl}(I-M)^{km} O^{nv} O^{ow}(\delta^{lm}\delta^{no} + \delta^{ln}\delta^{mo} + \delta^{lo}\delta^{mn})\right. \\
&\quad -\frac{1}{2} O^{ju} O^{mv} O^{nw} M^{kl}(\alpha^{jkl}\delta^{mn} + \alpha^{jkm}\delta^{ln} + \alpha^{jkn}\delta^{lm}) \\
&\quad \left. +\frac{\lambda+2}{6}\alpha^{jkl} O^{ju} O^{ov} O^{pw} M^{km} M^{ln}(\delta^{mn}\delta^{op} + \delta^{mo}\delta^{np} + \delta^{mp}\delta^{no})\right\} \\
&= n^{-2}\left\{\frac{1}{2} P^{ux}\theta^x_{jk}(I-M)^{jk}\delta^{vw} + \frac{\lambda-1}{6}\alpha^{jkl} O^{ju} M^{kl}\delta^{vw}\right. \\
&\quad \left. +\frac{\lambda-1}{3}\alpha^{jkl} O^{ju} O^{kv} O^{lw}\right\} + O(n^{-3}).
\end{aligned}
$$

Thus, the joint third-order cumulants are

$$
cum(R^u, R^v, R^w) = n^{-2}\lambda\alpha^{jkl} O^{ju} O^{kv} O^{lw} + O(n^{-3}).
$$

Note that this is $O(n^{-3})$ only when λ is $O(n^{-1})$; only empirical likelihood corrects for skewness properly.

Step 8. Joint fourth-order cumulants of R. The joint fourth-order cumulants of R are given (McCullagh, 1987, p.31) by

$$
\begin{aligned}
cum(R^u, R^v, R^w, R^x) &= E(R^u R^v R^w R^x) - E(R^u R^v)E(R^w R^x)[3] - E(R^u)E(R^v R^w R^x)[4] \\
&\quad +2E(R^u)E(R^v)E(R^w R^x)[6] - 6E(R^u)E(R^v)E(R^w)E(R^x).
\end{aligned}
$$

Again using the matching argument, it may be shown that all components having seven or more A terms are $O(n^{-4})$. Thus, the above expression can be reduced to

$$
\begin{aligned}
cum(R^u, R^v, R^w, R^x) &= E(R_1^u R_1^v R_1^w R_1^x) + E(R_2^u R_1^v R_1^w R_1^x)[4] + E(R_3^u R_1^v R_1^w R_1^x)[4] \\
&\quad +E(R_2^u R_2^v R_1^w R_1^x)[6] - E(R_1^u R_1^v)E(R_1^w R_1^x)[3] \\
&\quad -E(R_2^u R_1^v)E(R_1^w R_1^x)[12] - E(R_3^u R_1^v)E(R_1^w R_1^x)[12] \\
&\quad -E(R_2^u R_2^v)E(R_1^w R_1^x)[6] - E(R_2^u)E(R_1^v R_1^w R_1^x)[4] \\
&\quad -E(R_2^u)E(R_2^v R_1^w R_1^x)[12] + 2E(R_2^u)E(R_2^v)E(R_1^w R_1^x)[6] + O(n^{-4}).
\end{aligned}
$$

To save space, we introduce the following expressions (which will show up in the expectations). Let

$$
s_1 = \alpha^{jklm} O^{ju} O^{kv} O^{lw} O^{mx},
$$

$$
\begin{aligned}
s_2 &= \delta^{uv}\delta^{wx}+\delta^{uw}\delta^{vx}+\delta^{ux}\delta^{vw}, \\
s_3 &= \alpha^{jkl}\alpha^{mno}M^{no}(O^{ju}O^{kv}O^{lw}O^{mx}+O^{ju}O^{kv}O^{lx}O^{mw}+O^{ju}O^{kw}O^{lx}O^{mv} \\
&\quad +O^{jv}O^{kw}O^{lx}O^{mu}), \\
s_4 &= \alpha^{jkn}\alpha^{lmo}M^{no}(O^{ju}O^{kv}O^{lw}O^{mx}+O^{ju}O^{kw}O^{lv}O^{mx}+O^{ju}O^{kx}O^{lv}O^{mw}), \\
s_5 &= \theta^y_{mn}(I-M)^{mn}\alpha^{jkl}(P^{yu}O^{jv}O^{kw}O^{lx}+P^{yv}O^{ju}O^{kw}O^{lx}+P^{yw}O^{ju}O^{kv}O^{lx} \\
&\quad +P^{yx}O^{ju}O^{kv}O^{lw}.
\end{aligned}
$$

We now reduce the expectation above in a series of smaller steps. All terms of $O(n^{-4})$ are ignored.

Substep 1.

First,

$$
\begin{aligned}
E(R_1^uR_1^vR_1^wR_1^x) &= E(A^jA^kA^lA^mO^{ju}O^{kv}O^{lw}O^{mx}) \\
&= (n^{-2}-n^{-3})(\delta^{uv}\delta^{wx}+\delta^{uw}\delta^{vx}+\delta^{ux}\delta^{vw})+n^{-3}\alpha^{jklm}O^{ju}O^{kv}O^{lw}O^{mx} \\
&= (n^{-2}-n^{-3})s_2+n^{-3}s_1.
\end{aligned}
$$

Next,

$$E(R_1^uR_1^v)E(R_1^wR_1^x)=n^{-2}\delta^{uv}\delta^{wx},$$

so

$$E(R_1^uR_1^v)E(R_1^wR_1^x)[3]=n^{-2}s_2$$

and

$$E(R_1^uR_1^vR_1^wR_1^x)-E(R_1^uR_1^v)E(R_1^wR_1^x)[3]=n^{-3}(s_1-s_2).$$

Substep 2.

$$
\begin{aligned}
E(R_2^uR_1^v)E(R_1^wR_1^x) &= E[\frac{1}{2}P^{uy}\theta^y_{jk}(I-M)^{jl}(I-M)^{km}A^lA^mO^{nv}A^n \\
&\quad -\frac{1}{2}A^{jk}O^{ju}M^{kl}A^lO^{mv}A^m \\
&\quad +\frac{\lambda+2}{6}\alpha^{jkl}O^{ju}M^{km}M^{ln}A^mA^nO^{ov}A^o](n^{-1}\delta^{wx}) \\
&= n^{-3}(\frac{1}{2}P^{uy}\theta^y_{jk}(I-M)^{jl}(I-M)^{km}O^{nv}\alpha^{lmn} \\
&\quad -\frac{1}{2}O^{ju}M^{kl}O^{mv}\alpha^{jklm}+\frac{1}{2}\delta^{uv} \\
&\quad +\frac{\lambda+2}{6}\alpha^{jkl}O^{ju}M^{km}M^{ln}O^{ov}\alpha^{mno})\delta^{wx}.
\end{aligned}
$$

$$
\begin{aligned}
E(R_2^u R_1^v R_1^w R_1^x) &= E[\frac{1}{2}P^{uy}\theta^y_{jk}(I-M)^j l(I-M)^{km}O^{nv}O^{ow}O^{px}A^l A^m A^n A^o A^p \\
&\quad -\frac{1}{2}A^{jk}O^{ju}M^{kl}A^l O^{mv}O^{nw}O^{ox}A^m A^n A^o \\
&\quad +\frac{\lambda+2}{6}\alpha^{jkl}O^{ju}M^{km}A^m A^n O^{ov}O^{pw}O^{qx}A^o A^p A^q]
\end{aligned}
$$

We treat each of the three terms above separately. For the first term, we note that

$$
E(A^l A^m A^n A^o A^p) = n^{-3}\alpha^{lmn}\delta^{op}[10] + O(n^{-4}).
$$

Now, if either l or m appears in the δ term with any of the other three labels, then the product $O^{lu}(I-M)^{jl}$ will result, rendering that term null. The remaining terms yield

$$
\begin{aligned}
E(term\ 1) &= n^{-3}[\frac{1}{2}P^{uy}(I-M)^{jk}O^{nv}O^{ow}O^{pw}\alpha^{nop} \\
&\quad +\frac{1}{2}P^{uy}(I-M)^{jl}(I-M)^{km}(\alpha^{lmn}O^{nv}\delta^{wx})[3]].
\end{aligned}
$$

When cycled through $E(R_2^u R_1^v R_1^w R_1^x)[4]$, the first line in the equation above yields $n^{-3}\frac{1}{2}s_5$ and the second cancels the contribution from the first term of $E(R_2^u R_1^v)E(R_1^w R_1^x)[12]$. For the second term, we note that

$$
E(A^{jk}A^l A^m A^n A^o) = n^{-3}(\alpha^{jk*}\alpha^{***}[4] + (\alpha^{jk**} - \alpha^{jk}\alpha^{**})\alpha^{**}[6]) + O(n^{-4})
$$

so that

$$
\begin{aligned}
E(term\ 2) &= n^{-3}[-\frac{1}{2}O^{ju}M^{kl}O^{mv}O^{nw}O^{ox}\alpha^{jkl}\alpha^{mno} \\
&\quad -\frac{1}{2}O^{ju}M^{kl}O^{mv}O^{nw}O^{ox}(\alpha^{jkm}\alpha^{lno} + \alpha^{jkn}\alpha^{lmo} + \alpha^{jko}\alpha^{lmn}) \\
&\quad -\frac{1}{2}O^{ju}M^{kl}(\alpha^{jklm}O^{mv}\delta^{wx} + \alpha^{jkln}O^{nw}\delta^{vx} + \alpha^{jklo}O^{ox}\delta^{vw}) \\
&\quad -\frac{3}{2}\alpha^{jklm}O^{ju}O^{kv}O^{lw}O^{mx} \\
&\quad +(\delta^{uv}\delta^{wx} + \delta^{uw}\delta^{vx} + \delta^{ux}\delta^{vw})] + O(n^{-4}).
\end{aligned}
$$

Cycling through, the first line above yields $-n^{-3}\frac{1}{2}s_3$, the second line yields $-n^{-3}2s_4$, the third line cancels with the contribution from the first half of the second term of $E(R_2^u R_1^v)E(R_1^w R_1^x)[12]$, the fourth line yields $-n^{-3}6s_1$, and the last line yields $n^{-3}4s_2$, of which $2s_2$ cancel with the contribution from the second half of the second term of $E(R_2^u R_1^v)E(R_1^w R_1^x)[12]$. Now, for the third term,

$$\begin{aligned}
E(term\ 3) &= n^{-3}\frac{\lambda+2}{6}\alpha^{jkl}O^{ju}[\alpha^{opq}M^{kl}O^{ov}O^{pw}O^{qx} \\
&+M^{ln}\alpha^{nop}(O^{kv}O^{ow}O^{px}+O^{kw}O^{ov}O^{px}+O^{kx}O^{ov}O^{pw}) \\
&+M^{km}\alpha^{mop}(O^{lv}O^{ow}O^{px}+O^{lw}O^{ov}O^{px}+O^{lx}O^{ov}O^{pw}) \\
&+M^{km}M^{ln}(O^{qx}\alpha^{mnq}\delta^{vw}+O^{pw}\alpha^{mnw}\delta^{vx}+O^{ov}\alpha^{mnv}\delta^{wx})]+O(n^{-4}).
\end{aligned}$$

Cycling through, the first line above yields $n^{-3}\frac{\lambda+2}{6}s_3$, the second and third lines yield $n^{-3}\frac{4\lambda+8}{3}s_4$, and the the fourth line cancels with the contribution from the third term of $E(R_2^u R_1^v)E(R_1^w R_1^x)[12]$. Combining the above, we get

$$E(R_2^u R_1^v R_1^w R_1^x)[4]-E(R_2^u R_1^v)E(R_1^w R_1^x)[12]=n^{-3}(-6s_1+2s_2+\frac{\lambda-1}{6}s_3+\frac{4\lambda+2}{3}s_4+\frac{1}{2}s_5).$$

Substep 3.

Ignoring the $E(R_1^w R_1^x)=n^{-1}\delta^{wx}$ for now,

$$\begin{aligned}
E(R_2^u R_2^v) &= E[\frac{1}{4}P^{uy}P^{vz}\theta^y_{jk}\theta^z_{lm}(I-M)^{jn}(I-M)^{ko}(I-M)^{lp}(I-M)^{mq}A^nA^oA^pA^q \\
&-\frac{1}{4}P^{uy}\theta^y_{jk}(I-M)^{jl}(I-M)^{km}A^lA^mA^{no}O^{nv}M^{op}A^p \\
&-\frac{1}{4}P^{vy}\theta^y_{jk}(I-M)^{jl}(I-M)^{km}A^lA^mA^{no}O^{nu}M^{op}A^p \\
&+\frac{1}{4}A^{jk}A^{lm}O^{ju}O^{lv}M^{kn}M^{mo}A^nA^o \\
&+\frac{\lambda+2}{12}P^{uy}\theta^y_{jk}(I-M)^{jl}(I-M)^{km}A^lA^m\alpha^{nop}O^{nv}M^{oq}M^{pr}A^qA^r \\
&+\frac{\lambda+2}{12}P^{vy}\theta^y_{jk}(I-M)^{jl}(I-M)^{km}A^lA^m\alpha^{nop}O^{nu}M^{oq}M^{pr}A^qA^r \\
&-\frac{\lambda+2}{12}A^{jk}O^{ju}M^{kl}A^l\alpha^{mno}O^{mv}M^{np}M^{oq}A^pA^q \\
&-\frac{\lambda+2}{12}A^{jk}O^{jv}M^{kl}A^l\alpha^{mno}O^{mu}M^{np}M^{oq}A^pA^q \\
&+\frac{(\lambda+2)^2}{36}\alpha^{jkl}\alpha^{mno}O^{ju}O^{mv}M^{kp}M^{lq}M^{nr}M^{os}A^pA^qA^rA^s],
\end{aligned}$$

which becomes

$$\begin{aligned}
E(R_2^u R_2^v) &= n^{-2}[\frac{1}{4}P^{uy}P^{vz}\theta^y_{jk}\theta^z_{lm}((I-M)^{jk}(I-M)^{lm}+(I-M)^{jl}(I-M)^{km}+(I-M)^{jm}(I-M)^{kl}) \\
&-\frac{1}{4}P^{uy}\theta^y_{jk}(I-M)^{jk}O^{nv}M^{op}\alpha^{nop}
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{4}P^{vy}\theta^y_{jk}(I-M)^{jk}O^{nu}M^{op}\alpha^{nop}\\
&+\frac{1}{4}(\alpha^{jklm}O^{ju}O^{lv}M^{km}-\delta^{uv}+(\alpha^{jkn}\alpha^{lmo}+\alpha^{jko}\alpha^{lmn})O^{ju}O^{lv}M^{kn}M^{mo})\\
&+\frac{\lambda+2}{12}P^{uy}\theta^y_{jk}(I-M)^{jk}\alpha^{nop}O^{nv}M^{op}\\
&+\frac{\lambda+2}{12}P^{vy}\theta^y_{jk}(I-M)^{jk}\alpha^{nop}O^{nu}M^{op}\\
&-\frac{\lambda+2}{12}O^{ju}\alpha^{mno}O^{mv}\alpha^{jkl}(M^{kl}M^{no}+M^{ln}M^{ko}+M^{lo}M^{kn})\\
&-\frac{\lambda+2}{12}O^{jv}\alpha^{mno}O^{mu}\alpha^{jkl}(M^{kl}M^{no}+M^{ln}M^{ko}+M^{lo}M^{kn})\\
&\frac{(\lambda+2)^2}{36}\alpha^{jkl}\alpha^{mno}O^{ju}O^{mv}(M^{kl}M^{no}+M^{kn}M^{lo}+M^{ko}M^{ln})]+O(n^{-3}).
\end{aligned}
$$

Now,

$$
\begin{aligned}
E(R_2^uR_2^vR_1^wR_1^x) \;=\;& E[\frac{1}{4}P^{uy}P^{vz}\theta^y_{jk}\theta^z_{lm}(I-M)^{jn}(I-M)^{ko}(I-M)^{lp}(I-M)^{mq}A^nA^oA^pA^qO^{rw}A^rO^{sx}A^s\\
&-\frac{1}{4}P^{uy}\theta^y_{jk}(I-M)^{jl}(I-M)^{km}A^lA^mA^{no}O^{nv}M^{op}A^pO^{qw}A^qO^{rx}A^r\\
&-\frac{1}{4}P^{vy}\theta^y_{jk}(I-M)^{jl}(I-M)^{km}A^lA^mA^{no}O^{nu}M^{op}A^pO^{qw}A^qO^{rx}A^r\\
&+\frac{1}{4}A^{jk}A^{lm}O^{ju}O^{lv}M^{kn}M^{mo}A^nA^oO^{pw}A^pO^{qx}A^q\\
&+\frac{\lambda+2}{12}P^{uy}\theta^y_{jk}(I-M)^{jl}(I-M)^{km}A^lA^m\alpha^{nop}O^{nv}M^{oq}M^{pr}A^qA^rO^{sw}A^sO^{tx}A^t\\
&+\frac{\lambda+2}{12}P^{vy}\theta^y_{jk}(I-M)^{jl}(I-M)^{km}A^lA^m\alpha^{nop}O^{nu}M^{oq}M^{pr}A^qA^rO^{sw}A^sO^{tx}A^t\\
&-\frac{\lambda+2}{12}A^{jk}O^{ju}M^{kl}A^l\alpha^{mno}O^{mv}M^{np}M^{oq}A^pA^qO^{rw}A^rO^{sx}A^s\\
&-\frac{\lambda+2}{12}A^{jk}O^{jv}M^{kl}A^l\alpha^{mno}O^{mu}M^{np}M^{oq}A^pA^qO^{rw}A^rO^{sx}A^s\\
&+\frac{(\lambda+2)^2}{36}\alpha^{jkl}\alpha^{mno}O^{ju}O^{mv}M^{kp}M^{lq}M^{nr}M^{os}A^pA^qA^rA^sO^{hw}A^hO^{ix}A^i],
\end{aligned}
$$

which we will address term by term, as above.

$$E(term\,1)=n^{-3}\frac{1}{4}P^{uy}P^{vz}\theta^y_{jk}\theta^z_{lm}[(I-M)^{jk}(I-M)^{lm}+(I-M)^{jl}(I-M)^{km}+(I-M)^{jm}(I-M)^{kl}]\delta^{wx}+O(n^{-4})$$

which, when cycled, cancels the contribution from the first term of $E(R_2^uR_2^v)E(R_1^wR_1^x)$.

$$E(term\,2)=-n^{-3}\frac{1}{4}P^{uy}\theta^y_{jk}(I-M)^{jk}O^{nv}[\alpha^{nop}M^{op}\delta^{wx}+\alpha^{noq}O^{qw}O^{ox}+\alpha^{nor}O^{rx}O^{ow}]+O(n^{-4})$$

and

$$E(term\,3)=-n^{-3}\frac{1}{4}P^{vy}\theta^y_{jk}(I-M)^{jk}O^{nu}[\alpha^{nop}M^{op}\delta^{wx}+\alpha^{noq}O^{qw}O^{ox}+\alpha^{nor}O^{rx}O^{ow}]+O(n^{-4}).$$

Cycling, the first terms of these entries cancel with the contributions from the corresponding terms of $E(R_2^u R_2^v)E(R_1^w R_1^x)$; the second and third terms of both contribute a total of $-n^{-3}\frac{3}{2}s_5$.

$$\begin{aligned}
E(term\ 4) \quad = \quad & n^{-3}\frac{1}{4}[\alpha^{jklm}O^{ju}O^{lv}M^{km}\delta^{wx} - \delta^{uv}\delta^{wx} \\
& +\alpha^{jklm}O^{ju}O^{lv}O^{kw}O^{mx} - \delta^{uw}\delta^{vx} \\
& +\alpha^{jklm}O^{ju}O^{lv}O^{kx}O^{mw} - \delta^{ux}\delta^{vw} \\
& +\alpha^{jkn}\alpha^{lmo}O^{ju}O^{lv}M^{kn}M^{mo}\delta^{wx} \\
(5) \qquad & +\alpha^{jkn}\alpha^{lmp}O^{ju}O^{lv}O^{mx}M^{kn}O^{pw} \\
& +\alpha^{jkn}\alpha^{lmq}O^{ju}O^{lv}O^{mw}M^{kn}O^{qx} \\
& +\alpha^{jko}\alpha^{lmn}O^{ju}O^{lv}M^{kn}M^{mo}\delta^{wx} \\
& +\alpha^{jko}\alpha^{lmp}O^{ju}O^{lv}O^{kx}O^{pw}M^{mo} \\
& +\alpha^{jko}\alpha^{lmq}O^{ju}O^{lv}O^{kw}O^{qx}M^{mo} \\
(10) \qquad & +\alpha^{jkp}\alpha^{lmn}O^{ju}O^{lv}O^{mx}O^{pw}M^{kn} \\
& +\alpha^{jkp}\alpha^{lmo}O^{ju}O^{lv}O^{kx}O^{pw}M^{mo} \\
& +\alpha^{jkp}\alpha^{lmq}O^{ju}O^{lv}O^{pw}O^{qx}M^{km} \\
& +\alpha^{jkq}\alpha^{lmn}O^{ju}O^{lv}O^{mw}O^{qx}M^{kn} \\
& +\alpha^{jkq}\alpha^{lmo}O^{ju}O^{lv}O^{kw}O^{qx}M^{mo} \\
(15) \qquad & +\alpha^{jkq}\alpha^{lmp}O^{ju}O^{lv}O^{pw}O^{qx}M^{km}] + O(n^{-4}).
\end{aligned}$$

Cycling, the first, fourth and seventh lines together cancel the contribution from the corresponding term of $E(R_2^u R_2^v)E(R_1^w R_1^x)$. The second and third lines yield $n^{-3}(3s_1 - s_2)$. Lines five, six, eleven and fourteen yield $n^{-3}\frac{3}{2}s_3$. The remaining lines yield $n^{-3}3s_4$.

$$E(term\ 5) = n^{-3}\frac{\lambda+2}{12}P^{uy}\theta^y_{jk}(I-M)^{jk}\alpha^{nop}O^{nv}(M^{op}\delta^{wx} + O^{ow}O^{px} + O^{ox}O^{pw}) + O(n^{-4})$$

and

$$E(term\ 6) = n^{-3}\frac{\lambda+2}{12}P^{vy}\theta^y_{jk}(I-M)^{jk}\alpha^{nop}O^{nu}(M^{op}\delta^{wx} + O^{ow}O^{px} + O^{ox}O^{pw}) + O(n^{-4}).$$

The first terms of each of these cancel with the contibutions of the corresponding terms of

$E(R_2^u R_2^v)E(R_1^w R_1^x)$; the second and third terms of both contribute a total of $n^{-3}\frac{\lambda+2}{2}s_5$.

$$
\begin{aligned}
E(term\ 7) &= n^{-3}\frac{\lambda+2}{12}[\alpha^{jkl}O^{ju}M^{kl}\alpha^{mno}O^{mv}(M^{no}\delta^{wx}+O^{nw}O^{ox}(*)+O^{nx}O^{ow}(*)) \\
&+\alpha^{jkp}O^{ju}M^{np}\alpha^{mno}O^{mv}(M^{ko}\delta^{wx}+O^{kw}O^{ox}+O^{kx}O^{ow}) \\
&+\alpha^{jkq}O^{ju}M^{oq}\alpha^{mno}O^{mv}(M^{kn}\delta^{wx}+O^{kw}O^{nx}+O^{kx}O^{nw}) \\
&+\alpha^{jkr}O^{ju}O^{rw}\alpha^{mno}O^{mv}(M^{kn}O^{ox}+M^{ko}O^{nx}+O^{kx}M^{no}(*)) \\
&+\alpha^{jks}O^{ju}O^{sx}\alpha^{mno}O^{mv}(M^{kn}O^{ow}+M^{ko}O^{nw}+O^{kw}M^{no}(*))+O(n^{-4})
\end{aligned}
$$

and

$$
\begin{aligned}
E(term\ 8) &= n^{-3}\frac{\lambda+2}{12}[\alpha^{jkl}O^{jv}M^{kl}\alpha^{mno}O^{mu}(M^{no}\delta^{wx}+O^{nw}O^{ox}(*)+O^{nx}O^{ow}(*)) \\
&+\alpha^{jkp}O^{jv}M^{np}\alpha^{mno}O^{mu}(M^{ko}\delta^{wx}+O^{kw}O^{ox}+O^{kx}O^{ow}) \\
&+\alpha^{jkq}O^{jv}M^{oq}\alpha^{mno}O^{mu}(M^{kn}\delta^{wx}+O^{kw}O^{nx}+O^{kx}O^{nw}) \\
&+\alpha^{jkr}O^{jv}O^{rw}\alpha^{mno}O^{mu}(M^{kn}O^{ox}+M^{ko}O^{nx}+O^{kx}M^{no}(*)) \\
&+\alpha^{jks}O^{jv}O^{sx}\alpha^{mno}O^{mu}(M^{kn}O^{ow}+M^{ko}O^{nw}+O^{kw}M^{no}(*))+O(n^{-4}).
\end{aligned}
$$

The terms containing δ^{wx}, when cycled, serve to cancel the contributions from the corresponding terms of $E(R_2^u R_2^v)E(R_1^w R_1^x)$; the terms marked $(*)$ yield $-n^{-3}(\lambda+2)s_3$, and the remaining terms yield $-n^{-3}\frac{8}{3}(\lambda+2)s_4$.

$$
\begin{aligned}
E(term\ 9) &= n^{-3}\frac{(\lambda+2)^2}{36}\alpha^{jkl}\alpha^{mno}O^{ju}O^{mv}[M^{kl}M^{no}\delta^{wx}+M^{kl}O^{nw}O^{ox}(*)+M^{kl}O^{nx}O^{ow}(*) \\
&+M^{kn}M^{lo}\delta^{wx}+M^{kn}O^{lw}O^{ox}+M^{kn}O^{lx}O^{ow} \\
&+M^{ko}M^{ln}\delta^{wx}+M^{ko}O^{lw}O^{nx}+M^{ko}O^{lx}O^{nw} \\
&+O^{kw}M^{ln}O^{ox}+O^{kw}M^{lo}O^{nx}+O^{kw}O^{lx}M^{no}(*) \\
&+O^{kx}M^{ln}O^{ow}+O^{kx}M^{lo}O^{nw}+O^{kx}O^{lw}M^{no}(*)]+O(n^{-4}).
\end{aligned}
$$

The terms containing δ^{wx}, when cycled, cancel the contribution from the corresponding term of $E(R_2^u R_2^v)E(R_1^w R_1^x)$; the terms marked $(*)$ yield $n^{-3}\frac{(\lambda+2)^2}{6}s_3$, and the remaining terms yield $n^{-3}\frac{4}{9}(\lambda+2)^2 s_4$. Summing over all of the terms, we get

$$
E(R_2^u R_2^v R_1^w R_1^x)[6]-E(R_2^u R_2^v)E(R_1^w R_1^x)[6]=n^{-3}(3s_1-s_2+\frac{(\lambda-1)^2}{6}s_3+\frac{(2\lambda-5)(2\lambda+1)}{9}s_4+\frac{\lambda-1}{2}s_5).
$$

Substep 4.

Here we address the

$$E(R_3^u R_1^v R_1^w R_1^x)[4] - E(R_3^u R_1^v)E(R_1^w R_1^x)[12]$$

subterm. Note that there are three occurences of an A term in R_3, so that the only $O(n^{-3})$ terms arising from $E(R_3^u R_1^v R_1^w R_1^x)$ will be of the

$$E(A^1 A^2)E(A^3 A^4)E(A^5 A^6)[15]$$

variety. There are 60 such terms in all. On the other hand, noting that $E(R_1^w R_1^x)$ is $O(n^{-1})$, the only terms arising from $E(R_3^u R_1^v)$ which will contribute are of the form

$$E(A^1 A^2)E(A^3 A^4)[3].$$

There are 36 such terms, each of which cancels a corresponding term from $E(R_3^u R_1^v R_1^w R_1^x)$. The 24 remaining terms result from pairing each of the three A terms from R_3^u with an A term contributed from a different R_1 term. With this in mind, we proceed through the 17 terms of R_3^u given at the end of Step 5. The contributions from terms 1, 3, and 9-14 are all zero, in that all terms not cancelled give rise to a $(I-M)^{jk}O^{ju}$ component which vanishes. We proceed in order through the remaining 9 terms.

The noncancelled contribution from term 2 of $E(R_3^u R_1^v R_1^w R_1^x)$ is n^{-3} times

$$\begin{aligned} & \alpha^{jlo}\alpha^{kmp}M^{lm}O^{ju}O^{kx}O^{ov}O^{pw} + \alpha^{jlo}\alpha^{kmq}M^{lm}O^{ju}O^{kw}O^{ov}O^{qx} \\ + \; & \alpha^{jlp}\alpha^{kmo}M^{lm}O^{ju}O^{kx}O^{ov}O^{pw} + \alpha^{jlp}\alpha^{kmq}M^{lm}O^{ju}O^{kv}O^{pw}O^{qx} \\ + \; & \alpha^{jlq}\alpha^{kmo}M^{lm}O^{ju}O^{kw}O^{ov}O^{qx} + \alpha^{jlq}\alpha^{kmp}M^{lm}O^{ju}O^{kv}O^{pw}O^{qx} \end{aligned}$$

which, upon cycling, becomes $n^{-3}3s_4$.

Each of the 24 noncancelled terms from term 4 is of the form

$$n^{-3}\frac{\lambda+2}{6}\alpha^{jklm}O^{ju}O^{kv}O^{lw}O^{mx},$$

yielding $n^{-3}4(\lambda+2)s_1$.

The contributions from terms 5 and 6 both lead to pieces of s_4, so upon cycling the total is

$$24\ terms/((3\ terms)/s_4) \times -n^{-3}\frac{5}{12}(\lambda+2)$$

or $-n^{-3}\frac{10}{3}(\lambda+2)s_4$.

The contributions from term 7 again lead to s_4;

$$24\ terms/((3\ terms)/s_4) \times n^{-3}\frac{1}{9}(\lambda+2)^2$$

or $n^{-3}\frac{8}{9}(\lambda+2)^2 s_4$.

The contributions from term 8 lead to s_1;

$$24\ terms/((1\ term)/s_1) \times -n^{-3}\frac{(\lambda+2)(2\lambda+3)}{24}$$

or $-n^{-3}(\lambda+2)(2\lambda+3)s_1$.

The contributions from terms 15, 16 and 17 all lead to s_2;

$$24\ terms/((3\ terms)/s_2) \times n^{-3}\frac{\lambda^2}{8}$$

or $n^{-3}\lambda^2 s_2$.

Thus,

$$E(R_3^u R_1^v R_1^w R_1^x)[4] - E(R_3^u R_1^v)E(R_1^w R_1^x)[12] = -(\lambda+2)(2\lambda-1)s_1 + \lambda^2 s_2 + \frac{8\lambda^2+2\lambda-1}{9}s_4.$$

Substep 5.

Note that the expression

$$-E(R_2^u)E(R_1^v R_1^w R_1^x)[4] - E(R_2^u)E(R_2^v R_1^w R_1^x)[12] + 2E(R_2^u)E(R_2^v)E(R_1^w R_1^x)[6]$$

can be rewritten as

$$-E(R_2^u)cum(R^v, R^w, R^x)[4].$$

Then, as

$$\begin{aligned} E(R_2^u)cum(R^v, R^w, R^x) &= n^{-3}[\frac{1}{2}P^{uy}\theta^y_{jk}(I-M)^{jk}\lambda\alpha^{lmn}O^{lv}O^{mw}O^{nx} \\ &\quad + \frac{\lambda-1}{6}\alpha^{jkl}O^{ju}M^{kl}\lambda\alpha^{mno}O^{mv}O^{nw}O^{ox}] + O(n^{-4}), \end{aligned}$$

we get

$$-E(R_2^u)cum(R^v, R^w, R^x)[4] = -n^{-3}(\frac{\lambda(\lambda-1)}{6}s_3 + \frac{\lambda}{2}s_5).$$

Combining the results of the substeps above, the fourth order joint cumulants are

$$
\begin{array}{ccccc}
s_1 & -s_2 & & & \\
-6s_1 & 2s_2 & \frac{\lambda-1}{6}s_3 & \frac{4\lambda+2}{3}s_4 & \frac{1}{2}s_5 \\
3s_1 & -s_2 & \frac{(\lambda-1)^2}{6}s_3 & \frac{(2\lambda-5)(2\lambda+1)}{9}s_4 & \frac{\lambda-1}{2}s_5 \\
-(\lambda+2)(2\lambda-1)s_1 & \lambda^2 s_2 & & \frac{8\lambda^2+2\lambda-1}{9}s_4 & \\
 & & -\frac{\lambda(\lambda-1)}{6}s_3 & & -\frac{\lambda}{2}s_5 \\
\hline
-\lambda(2\lambda+3)s_1 & \lambda^2 s_2 & & \frac{1}{3}\lambda(4\lambda+2)s_4 &
\end{array}
$$

times n^{-3}. These only vanish when $\lambda = 0$.

Step 9. Approximation to the distribution of CR_0. Let the mean vector and the covariance matrix of $n^{1/2}R$ be denoted by $\delta = (\delta^1, \ldots, \delta^q)^T$ and Δ, respectively. Then δ is $O(n^{-1/2})$ and $\Delta = I + \Delta_1 + O(n^{-2})$, where Δ_1 is $O(n^{-1})$. It has been shown in Steps 7 and 8 that the joint third and fourth-order cumulants of $n^{1/2}R$ are $O(n^{-1/2})$ and $O(n^{-1})$ in general, and $O(n^{-3/2})$ and $O(n^{-2})$ when $\lambda = 0$; furthermore, the matching arguments shown there can be used to show that the higher order joint cumulants are all $O(n^{-3/2})$ or smaller. Thus, to error of order $O(n^{-1/2})$, or $O(n^{-3/2})$ when $\lambda = 0$, the quantity nR^TR has a multivariate normal distribution.

It follows then that the sth cumulant of nR^TR is equal to the sth cumulant of a χ^2_q distribution plus an error term; this error term is $O(n^{-3/2})$ when $\lambda = 0$.

Step 10. Expansion of $E(nR^TR)$. Recall from Step 5 that

$$R^uR^u = R_1^uR_1^u + 2R_1^uR_2^u + 2R_1^uR_3^u + R_2^uR_2^u + O_p(n^{-5/2}).$$

Thus,

$$E(R^uR^u) = E(R_1^uR_1^u) + 2E(R_1^uR_2^u) + 2E(R_1^uR_3^u) + E(R_2^uR_2^u) + O(n^{-3}).$$

Now, define

$$
\begin{aligned}
t_1 &= \alpha^{jkl}\alpha^{mno}M^{jm}M^{kn}M^{lo}, \\
t_2 &= \alpha^{jkl}\alpha^{mno}M^{jk}M^{lm}M^{no}, \\
t_3 &= \alpha^{jklm}M^{jk}M^{lm}, \\
t_4 &= \alpha^{jkl}N^{ju}\theta^u_{mn}(I-M)^{mk}(I-M)^{nl}, \\
t_5 &= Q^{uv}\theta^u_{jk}\theta^v_{lm}\{(I-M)^{jk}(I-M)^{lm} + 2(I-M)^{jl}(I-M)^{km}\}, \\
t_6 &= \alpha^{jkl}N^{ju}\theta^u_{mn}(I-M)^{mn}M^{kl},
\end{aligned}
$$

where $Q = (\Theta\Theta^T)^{-1}$ (these will be used to simplify the algebra shortly). Recall that $M = \Theta^T(\Theta\Theta^T)^{-1}\Theta$ and $N = \Theta^T(\Theta\Theta^T)^{-1}$ are defined in Step 4. We now expand the four expectation terms in terms of this notation below. For the first term,

$$E(R_1^u R_1^u) = n^{-1}\delta^{uu} = n^{-1}q.$$

For the second term,

$$\begin{aligned}
E(R_1^u R_2^u) &= E[\frac{1}{2}P^{uy}\theta^y_{jk}(I-M)^{jl}(I-M)^{km}A^lA^mO^{nu}A^n \\
&\quad -\frac{1}{2}A^{jk}O^{ju}M^{kl}A^lO^{mu}A^m \\
&\quad +\frac{\lambda+2}{6}\alpha^{jkl}O^{ju}M^{km}M^{ln}A^mA^nO^{ou}A^o] \\
&= n^{-2}(\frac{1}{2}N^{ny}\theta^y_{jk}(I-M)^{jl}(I-M)^{km}\alpha^{lmn} \\
&\quad -\frac{1}{2}\alpha^{jklm}M^{kl}M^{lm}+\frac{1}{2}\delta^{uu} \\
&\quad +\frac{\lambda+2}{6}\alpha^{jkl}\alpha^{mno}M^{km}M^{ln}M^{jo}) + O(n^{-3}) \\
&= n^{-2}(\frac{\lambda+2}{6}t_1 - \frac{1}{2}t_3 + \frac{1}{2}t_4 + \frac{1}{2}q) + O(n^{-3}).
\end{aligned}$$

For the third term,

$$\begin{aligned}
E(R_1^u R_3^u) &= E[-\frac{1}{4}O^{ju}N^{kv}\theta^v_{lm}(I-M)^{ln}(I-M)^{mo}A^{jk}A^nA^oO^{pu}A^p \\
&\quad +\frac{3}{8}O^{ju}M^{lm}M^{kn}A^{jl}A^{km}A^nO^{ou}A^o \\
&\quad +\frac{\lambda+2}{6}O^{ju}\alpha^{jkl}M^{km}N^{lv}\theta^v_{no}(I-M)^{np}(I-M)^{oq}A^mA^pA^qO^{ru}A^r \\
&\quad +\frac{\lambda+2}{6}O^{ju}M^{km}M^{ln}A^{jkl}A^mA^nO^{ou}A^o \\
(5) \qquad &\quad -c_1\frac{5}{12}(\lambda+2)O^{ju}\alpha^{jkm}M^{mn}M^{ko}M^{lp}A^{ln}A^oA^pO^{qu}A^q \\
&\quad -(1-c_1)\frac{5}{12}(\lambda+2)O^{ju}\alpha^{kln}M^{mn}M^{ko}M^{lp}A^{jm}A^oA^pO^{qu}A^q \\
&\quad +\frac{(\lambda+2)^2}{9}O^{ju}\alpha^{jkn}\alpha^{lmo}M^{no}M^{kp}M^{lq}M^{mr}A^pA^qA^rO^{su}A^s \\
&\quad -\frac{(\lambda+2)(2\lambda+3)}{24}O^{ju}\alpha^{jklm}M^{kn}M^{lo}M^{mp}A^nA^oA^pO^{qu}A^q \\
&\quad -\frac{1}{2}P^{uv}\theta^v_{jk}N^{jw}\theta^w_{mn}(I-M)^{kl}(I-M)^{mo}(I-M)^{np}A^lA^oA^pO^{qu}A^q \\
(10) \qquad &\quad -c_2P^{uv}\theta^v_{jk}(I-M)^{jl}(I-M)^{km}M^{no}A^lA^{mn}A^oO^{pu}A^p \\
&\quad -(1-c_2)O^{ju}N^{nv}\theta^v_{kl}(I-M)^{km}(I-M)^{lo}M^{np}A^{jm}A^oA^pO^{qu}A^q \\
&\quad +c_3\frac{(\lambda+2)}{2}P^{uv}\theta^v_{jk}(I-M)^{jl}(I-M)^{km}\alpha^{mno}M^{np}M^{oq}A^lA^pA^qO^{ru}A^r
\end{aligned}$$

$$
\begin{aligned}
&+(1-c_3)\frac{(\lambda+2)}{2}O^{ju}N^{nv}\theta^v_{kl}(I-M)^{km}(I-M)^{lp}\alpha^{jmo}M^{nq}M^{or}A^pA^qA^rO^{su}A^s \\
&+\frac{1}{6}P^{uv}\theta^v_{jkl}(I-M)^{jm}(I-M)^{kn}(I-M)^{lo}A^mA^nA^oO^{pu}A^p \\
(15)\qquad &+\frac{\lambda^2+4\lambda}{8}O^{ju}M^{jl}M^{km}M^{kn}A^lA^mA^nO^{ou}A^o \\
&-c_4\frac{\lambda}{2}O^{ju}M^{km}M^{kn}A^jA^mA^nO^{ou}A^o \\
&-(1-c_4)\frac{\lambda}{2}O^{ku}M^{jl}M^{km}A^jA^lA^mO^{nu}A^n].
\end{aligned}
$$

We will evaluate this line by line, ignoring terms of $O(n^{-3})$.

Line 1 becomes

$$-n^{-2}\frac{1}{4}M^{jp}N^{kv}\theta^v_{lm}(I-M)^{ln}(I-M)^{mo}[\alpha^{jkn}\delta^{op}+\alpha^{jko}\delta^{np}+\alpha^{jkp}\delta^{no}].$$

The first two of the above terms involve $(I-M)M$ terms and are null, and the last yields

$$-n^{-2}\frac{1}{4}\alpha^{jkp}M^{jp}N^{kv}\theta^v_{lm}(I-M)^{lm},$$

or $-n^{-2}\frac{1}{4}t_6$.

Line 2 reduces to

$$n^{-2}\frac{3}{8}M^{jo}M^{lm}M^{kn}[(\alpha^{jklm}\delta^{no}-\delta^{jl}\delta^{km}\delta^{no})+\alpha^{jln}\alpha^{kmo}+\alpha^{jlo}\alpha^{kmn}],$$

which reduces to

$$n^{-2}\frac{3}{8}(t_1+t_2+t_3-q).$$

Line 3 becomes

$$n^{-2}\frac{\lambda+2}{6}M^{jr}\alpha^{jkl}M^{km}N^{lv}\theta^v_{no}(I-M)^{np}(I-M)^{oq}[\delta^{mp}\delta^{qr}+\delta^{mq}\delta^{pr}+\delta^{mr}\delta^{pq}].$$

The first two of the above terms involve $(I-M)M$ terms and are null, and the last yields

$$n^{-2}\frac{\lambda+2}{6}M^{jk}\alpha^{jkl}N^{lv}\theta^v_{no}(I-M)^{no},$$

or $n^{-2}\frac{1}{6}(\lambda+2)t_6$.

Line 4 becomes

$$n^{-2}\frac{\lambda+2}{6}M^{jo}M^{km}M^{ln}[\alpha^{jklm}\delta^{no}+\alpha^{jkln}\delta^{mo}+\alpha^{jklo}\delta^{mn}],$$

which reduces to $n^{-2}\frac{1}{2}(\lambda+2)t_3$.

Line 5 becomes

$$-n^{-2}c_1\frac{5}{12}(\lambda+2)\alpha^{jkm}M^{jq}M^{mn}M^{ko}M^{lp}[\alpha^{lno}\delta^{pq}+\alpha^{lnp}\delta^{oq}+\alpha^{lnq}\delta^{op}].$$

The first and third of the above terms yield t_1 components (each of the three indices of a given α term is associated with a different M term) and the second yields a t_2 component, so this reduces to $-n^{-2}c_1\frac{5}{12}(\lambda+2)(2t_1+t_2)$.

Line 6 becomes

$$-n^{-2}(1-c_1)\frac{5}{12}(\lambda+2)\alpha^{kln}M^{jq}M^{mn}M^{ko}M^{lp}[\alpha^{jmo}\delta^{pq}+\alpha^{jmp}\delta^{oq}+\alpha^{jmq}\delta^{op}].$$

The first and second of the above terms yield t_1 components and the third yields a t_2 component, so this reduces to $-n^{-2}(1-c_1)\frac{5}{12}(\lambda+2)(2t_1+t_2)$.

Line 7 becomes

$$n^{-2}\frac{(\lambda+2)^2}{9}\alpha^{jkn}\alpha^{lmo}M^{no}M^{kp}M^{js}M^{lq}M^{mr}[\delta^{pq}\delta^{rs}+\delta^{pr}\delta^{qs}+\delta^{ps}\delta^{qr}],$$

which reduces to

$$n^{-2}\frac{(\lambda+2)^2}{9}\alpha^{jkn}\alpha^{lmo}M^{no}[M^{jm}M^{kl}+M^{km}M^{jl}+M^{jk}M^{lm}].$$

The first and second of the above terms yield t_1 components and the third yields a t_2 component, so this reduces to $n^{-2}\frac{1}{9}(\lambda+2)^2(2t_1+t_2)$.

Line 8 becomes

$$-n^{-2}\frac{(\lambda+2)(2\lambda+3)}{24}\alpha^{jklm}M^{jq}M^{kn}M^{lo}M^{mp}[\delta^{no}\delta^{pq}+\delta^{np}\delta^{oq}+\delta^{nq}\delta^{op}],$$

which reduces to $-n^{-2}\frac{1}{8}(\lambda+2)(2\lambda+3)t_3$.

Line 9 becomes

$$-n^{-2}\frac{1}{2}N^{qv}\theta^v_{jk}N^{jw}\theta^w_{mn}(I-M)^{kl}(I-M)^{mo}(I-M)^{np}[\delta^{lo}\delta^{pq}+\delta^{lp}\delta^{oq}+\delta^{lq}\delta^{op}].$$

Each of the above terms has an $(I-M)N$ component and is hence null; the contribution of this line is 0.

Line 10 becomes

$$-n^{-2}c_2N^{pv}\theta^v_{jk}(I-M)^{jl}(I-M)^{km}M^{no}[\alpha^{lmn}\delta^{op}+\alpha^{mno}\delta^{lp}+\alpha^{mnp}\delta^{lo}].$$

The second and third of the above terms have $(I-M)N$ and $(I-M)M$ components, respectively, and are hence null; the first yields

$$-n^{-2}c_2N^{nv}\theta^v_{jk}(I-M)^{jl}(I-M)^{km}\alpha^{lmn}$$

or $-n^{-2}c_2t_4$.

Line 11 becomes

$$-n^{-2}(1-c_2)N^{nv}\theta^v_{kl}(I-M)^{km}(I-M)^{lo}M^{jq}M^{np}[\alpha^{jmo}\delta^{pq}+\alpha^{jmp}\delta^{oq}+\alpha^{jmq}\delta^{op}].$$

The second and third of the above terms have $(I-M)M$ components, and are hence null; the first yields

$$-n^{-2}(1-c_2)N^{jv}\theta^v_{kl}(I-M)^{km}(I-M)^{lo}\alpha^{jmo}$$

or $-n^{-2}(1-c_2)t_4$.

Line 12 becomes

$$n^{-2}c_3\frac{\lambda+2}{2}N^{rv}\theta^v_{jk}(I-M)^{jl}(I-M)^{km}\alpha^{mno}M^{np}M^{oq}[\delta^{lp}\delta^{qr}+\delta^{lq}\delta^{pr}+\delta^{lr}\delta^{pq}].$$

The first and second of the above terms have $(I-M)M$ components, and the third has an $(I-M)N$ component, so the total contribution from this line is 0.

Line 13 becomes

$$n^{-2}(1-c_3)\frac{\lambda+2}{2}N^{nv}\theta^v_{kl}(I-M)^{km}(I-M)^{lp}\alpha^{jmo}M^{nq}M^{or}M^{js}[\delta^{pq}\delta^{rs}+\delta^{pr}\delta^{qs}+\delta^{ps}\delta^{qr}].$$

All of the above terms have $(I-M)M$ components, so the total contribution from this line is 0.

Line 14 becomes

$$n^{-2}\frac{1}{6}N^{pv}\theta^v_{jkl}(I-M)^{jm}(I-M)^{kn}(I-M)^{lo}[\delta^{mn}\delta^{op}+\delta^{mo}\delta^{np}+\delta^{mp}\delta^{no}].$$

All of the above terms have $(I-M)M$ components, so the total contribution from this line is 0.

Line 15 becomes

$$n^{-2}\frac{\lambda^2+4\lambda}{8}M^{lo}M^{mn}[\delta^{lm}\delta^{no}+\delta^{ln}\delta^{mo}+\delta^{lo}\delta^{mn}]$$

which reduces to $n^{-2}\frac{1}{8}(\lambda^2+4\lambda)(q^2+2q)$.

Line 16 becomes

$$-n^{-2}c_4\frac{\lambda}{2}M^{jo}M^{mn}[\delta^{jm}\delta^{no}+\delta^{jn}\delta^{mo}+\delta^{jo}\delta^{mn}]$$

which reduces to $-n^{-2}c_4\frac{1}{2}\lambda(q^2+2q)$.

Line 17 becomes

$$-n^{-2}(1-c_4)\frac{\lambda}{2}M^{jl}M^{mn}[\delta^{jl}\delta^{mn}+\delta^{jm}\delta^{ln}+\delta^{jn}\delta^{lm}]$$

which reduces to $-n^{-2}(1-c_4)\frac{1}{2}\lambda(q^2+2q)$.

Summing these contributions, we get

$$n^{-2}[\frac{1}{72}(16\lambda^2+4\lambda-29)t_1+\frac{1}{72}(8\lambda^2+2\lambda-1)t_2+\frac{1}{8}(-2\lambda^2-3\lambda+5)t_3-t_4 \\ +\frac{1}{12}(2\lambda+1)t_6+\frac{1}{8}(2\lambda^2-3)q+\frac{1}{8}\lambda^2q^2].$$

Note that when $\lambda=0$, this does not reduce to the equation given in DHR. The t_1 and t_2 entries given in DHR are $43/72$ and $-73/72$; these are 1 more and 1 less than those given here, respectively.

For the fourth term, $E(R_2^uR_2^u)$, the expansion can readily be found by examining the expansion of $E(R_2^uR_2^v)$ given in Substep 3 of Step 8. The nine lines of the expansion given there yield, respectively, $n^{-2}\frac{1}{4}t_5$, $-n^{-2}\frac{1}{4}t_6$, $-n^{-2}\frac{1}{4}t_6$, $n^{-2}\frac{1}{4}(t_3-q+t_1+t_2)$, $n^{-2}\frac{1}{12}(\lambda+2)t_6$, $n^{-2}\frac{1}{12}(\lambda+2)t_6$, $-n^{-2}\frac{1}{12}(\lambda+2)(2t_1+t_2)$, $-n^{-2}\frac{1}{12}(\lambda+2)(2t_1+t_2)$, and $n^{-2}\frac{1}{36}(\lambda+2)^2(2t_1+t_2)$. When summed, this becomes

$$n^{-2}[\frac{1}{36}(2\lambda^2-4\lambda-7)t_1+\frac{1}{36}(\lambda^2-2\lambda+1)t_2+\frac{1}{4}t_3+\frac{1}{4}t_5+\frac{1}{6}(\lambda-1)t_6-\frac{1}{4}q].$$

Therefore,

$$\begin{aligned} E(nR^TR) &= nE(R^uR^u) \\ &= q+n^{-1}[\frac{1}{6}(3\lambda^2+2\lambda-2)t_1+\frac{1}{4}\lambda^2t_2+\frac{1}{4}(-2\lambda^2-3\lambda+2)t_3-t_4 \\ &\quad +\frac{1}{4}t_5+\frac{1}{2}\lambda t_6+\frac{1}{2}\lambda^2q+\frac{1}{4}\lambda^2q^2]+O(n^{-2}). \end{aligned}$$

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