Stat 310 Homework 2 Key

Chapter 2, problems 2, 10, 11, 20, 23, 26, 40, 45, 47, 52. Due 9/16/99.

2.2. An experiment consists of throwing a fair coin 4 times. Find the frequency function and the cumulative distribution function for the follwing random variables: (a) the number of heads before the first tail, (b) the number of heads following the first tail, (c) the number of heads minus the number of tails, and (d) the number of tails times the number of heads.

Outcome	X_a	X_b	X_c	X_d
hhhh	4	0	4	0
hhht	3	0	2	3
hhth	2	1	2	3
hhtt	2	0	0	4
hthh	1	2	2	3
htht	1	1	0	4
htth	1	1	0	4
httt	1	0	-2	3
thhh	0	3	2	3
thht	0	2	0	4
thth	0	2	0	4
thtt	0	1	-2	3
tthh	0	2	0	4
ttht	0	1	-2	3
ttth	0	1	-2	3
tttt	0	0	-4	0

For all parts of this problem, we can find the frequency functions by enumerating the space of equally likely outcomes and counting. The cumulative distribution functions can then be found by summing the frequency functions. Other ways of solving each case are given below.

a) Enumerating,

X_a	0	1	2	3	4
p_{X_a}	8/16	4/16	2/16	1/16	1/16
F_{X_a}	8/16	12/16	14/16	15/16	1

Here we take the case of four heads as giving $X_a = 4$. We can also solve this one by noting that if Y denotes the number of the trial on which the first tail occurs, then $X_a = Y - 1$ when Y is in the range $[1, 4]$. Now, the distribution of Y is geometric with $p = 1/2$ (a fair coin), so $P(Y = k) = pq^{k-1}$ leads to probabilities of 1/2, 1/4, 1/8 and 1/16. The result for $X = 4$ corresponds to Y taking on any value greater than 4, and we can find the probability of this by simply making sure that all of the probabilities add up to one.

b) Enumerating,

X_b	0	1	2	3
p_{X_b}	5/16	6/16	4/16	1/16
F_{X_b}	5/16	11/16	15/16	1

Here we take the case of four heads as giving $X_b = 0$ (the first tail never occurred, so we never started counting heads).

c) Enumerating,

X_c	-4	-2	0	2	4
p_{X_c}	1/16	4/16	6/16	4/16	1/16
F_{X_c}	1/16	5/16	11/16	15/16	1

We can also solve this one by noting that if Y represents the total number of heads, then the number of tails is $4 - Y$, and the number of heads minus the number of tails is $2Y - 4$. As the random variable Y has a Binomial(6,1/2) distribution, the frequency function for X_c follows.

d) Enumerating,

X_d	0	3	4
p_{X_d}	6/16	8/16	2/16
F_{X_d}	6/16	14/16	1

We can also solve this one by noting that if Y represents the total number of heads, then the number of tails is $4 - Y$, and the number of heads times the number of tails is $4Y - Y^2$. As the random variable Y has a Binomial(6,1/2) distribution, the frequency function for X_d follows.

2.10. Appending 3 extra bits to a 4-bit word in a particular way (a Hamming code) allows detection and correction of up to one error in any of the bits. If each bit has probability .05 of being changed during communication, and the bits are changed independently of each other, what is the probability that the word is correctly received (that is, 0 or 1 bit is in error)?

Let X denote the number of bits in error. Then the probability of correct word transmission is

$$P(X = 0) + P(X = 1).$$

The distribution of X is Binomial(7,.05), so this becomes

$$\binom{7}{0}(.05)^0(.95)^7 + \binom{7}{1}(.05)^1(.95)^6 = 0.6983 + 0.2573 = 0.9556.$$

How does this probability compare to the probability that the word will be transmitted correctly with no check bits, in which case all four bits would have to be transmitted correctly for the word to be correct?

Again, let X denote the number of bits in error. In this case, the distribution of X is again Binomial, but the parameter n has changed from 7 to 4. Thus, the chance of correct word transmission is

$$P(X = 0) = \binom{4}{0}(.05)^0(.95)^4 = 0.8145.$$

This is less than the probability of correct transmission using Hamming bits, so the encoding helps here.

2.11. Consider the binomial distribution with n trials and probability p of success on each trial. For what value of k is $P(X = k)$ maximized? This value is called the **mode** of the distribution. (*Hint:* Consider the ratio of successive terms.)

Ok, we begin by examining the expressions for $P(X = k)$ and $P(X = k+1)$, with an eye towards writing the result as

$$P(X = k) * c = P(X = k + 1).$$

Expanding,

$$\begin{aligned}
\binom{n}{k} p^k (1-p)^{n-k} c &= \binom{n}{k+1} p^{k+1} (1-p)^{n-k-1} \\
\frac{n!}{k!(n-k)!}(1-p)c &= \frac{n!}{(k+1)!(n-k-1)!}p \\
\frac{(1-p)}{n-k}c &= \frac{p}{k+1} \\
c &= \frac{(n-k)p}{(k+1)(1-p)}.
\end{aligned}$$

Note that c is monotonically decreasing as k increases. As long as c is greater than 1, the probability increases if we move to the succeeding term. We can also move on if c equals 1; there will be a decrease beyond that point. So, starting at $k = 0$, we keep moving to the right as long as

$$\begin{aligned}
1 &\leq \frac{(n-k)p}{(k+1)(1-p)} \\
(k+1)(1-p) &\leq (n-k)p \\
k &\leq np - (1-p).
\end{aligned}$$

As k is an integer, the last k for which we want to increase one step is given by the integer part of the right-hand side expression, so the mode is at a value one greater than that, namely

$$\lfloor np - (1-p) \rfloor + 1 = \lfloor np - (1-p) + 1 \rfloor = \lfloor (n+1)p \rfloor.$$

2.20. If X is a geometric random variable with $p = .5$, for what value of k is $P(X \leq k) \approx .99$?

This is actually asking us for k such that $F_X(k) \approx .99$, so it behooves us to try to come up with a formula for the cdf of a geometric random variable. By definition,

$$\begin{aligned}
P(X \leq k) &= \sum_{i=1}^{k} pq^{i-1} \\
&= \sum_{i=1}^{\infty} pq^{i-1} - \sum_{j=k+1}^{\infty} pq^{j-1}.
\end{aligned}$$

Now, the first infinite sum is simply the chance that a geometric random variable will take on any value at all, and hence is 1. Bringing out a common factor, we can rewrite the second sum over a different summation range as

$$pq^k \sum_{i=0}^{\infty} q^i = pq^k \frac{1}{1-q} = q^k,$$

so $F_X(k) = 1 - q^k$. (Quick check - plugging in values for $k = 1$ and $k = 2$ confirm that this works there.) Thus,

$$\begin{aligned} 1 - q^k &\approx 0.99 \\ q^k &\approx 0.01 \\ k &\approx \log(0.01)/\log(q). \end{aligned}$$

As $q = 1 - p = 0.5$, this yields $k \approx 6.64$. Checking in this neighborhood, we find that $F_X(6) = 0.9844$, and $F_X(7) = 0.9922$, so we choose $k = 7$ as the closest.

2.23. In a sequence of independent trials with probability p of success, what is the probability that there are r successes before the kth failure?

Well, this can be thought of as a negative binomial situation in that we are waiting for the kth occurrence of something. In this case, we know that there are a $r + k$ total trials, the last of which is a failure, and that there are $\binom{r+k-1}{r}$ ways of selecting the positions at which the r successes can occur in the first $r + k - 1$ trials, so the probability is

$$\binom{r+k-1}{r} p^r (1-p)^k.$$

2.26. The university administration assures a mathematician that he has only one chance in 10000 of being trapped in a much-maligned elevator in the mathematics building. If he goes to work 5 days a week, 52 weeks a year, for 10 years and always rides the elevator up to his office when he first arrives, what is the chance that he will never be trapped? That he will be trapped once? Twice? Assume that the outcomes on all the days are mutually independent (a dubious assumption in practice).

Let X denote the number of times that the mathematician is trapped in the elevator. The distribution of X is binomial with parameters $n = 5 * 52 * 10 = 2600$ and $p = 0.0001$. As n is quite large and p is quite small, we can accurately approximate the distribution of X with a Poisson distribution having parameter $\lambda = np = 0.26$. With this approximation,

$$\begin{aligned} P(X = 0) &= \frac{(0.26)^0 e^{-.26}}{0!} = .7711 \\ P(X = 1) &= \frac{(0.26)^1 e^{-.26}}{1!} = .2005 \\ P(X = 2) &= \frac{(0.26)^2 e^{-.26}}{2!} = .0261 \end{aligned}$$

2.40. Suppose that X has the density function $f(x) = cx^2$ for $0 \le x \le 1$ and $f(x) = 0$ otherwise.

a) Find c. Well, we know that the density function must integrate to 1, so

$$\int_{-\infty}^{\infty} f(x)dx = \int_0^1 cx^2 = \left.\frac{cx^3}{3}\right|_0^1 = c/3,$$

and $c = 3$.

b) Find the cdf. Well, by definition

$$F_X(x) = \int_{-\infty}^{x} f_X(u)du,$$

so, using the integral found above,

$$F_X(x) = \begin{cases} 0 & x < 0 \\ x^3 & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$$

c) What is $P(.1 \le X \le .5)$? This is simply $F_X(.5) - F_X(.1) = .125 - .001 = .124$.

2.45. Suppose that the lifetime of an electronic component follows an exponential distribution with $\lambda = .1$.

a) Find the probability that the lifetime is less than 10. Letting X denote the lifetime, we know that the cdf $F_X(x) = 1 - e^{-\lambda x}$, $x \ge 0$. Thus,

$$F_X(10) = 1 - \exp(-1) = 0.6321.$$

b) Find the probability that the lifetime is between 5 and 15.

$$F_X(15) - F_X(5) = \exp(-0.5) - \exp(-1.5) = .6065 - .2231 = .3834.$$

2.47. If $\alpha > 1$, show that the gamma density has a maximum at $(\alpha - 1)/\lambda$.

This is a fairly straightforward exercise in differentiation. The gamma density is given by

$$\frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}.$$

Differentiating with respect to x gives

$$\frac{\lambda^\alpha}{\Gamma(\alpha)} \left[(\alpha - 1)x^{\alpha-2} - \lambda x^{\alpha-1}\right] e^{-\lambda x},$$

and setting the term in square brackets to zero gives

$$(\alpha - 1) - \lambda x = 0; \quad x = (\alpha - 1)/\lambda.$$

As the term in square brackets also determines the sign of the derivative, we can see that it is positive to the left of the critical point and negative to the right, so this is indeed a maximum.

2.52. Suppose that in a certain population, individual's heights are approximately normally distributed with parameters $\mu = 70$ and $\sigma = 3$in.

a) What proportion of the population is over 6 ft. tall? Well, letting X be the height in inches, we are being asked for $P(X > 72)$. To find this, we need to map X over to a standard normal; $Y = (X - \mu)/\sigma$ has a $N(0, 1)$ distribution. Thus,

$$\begin{aligned} P(X > 72) &= P\left(\frac{X - 70}{3} > \frac{72 - 70}{3}\right) \\ &= P(Y > 0.67) \\ &= 1 - \Phi(0.67) \\ &= 1 - .7486 = 0.2514. \end{aligned}$$

The value for $\Phi(0.67)$ was obtained from Table 2 of Appendix B of your text, on page A7. Alternatively, it can be found in MATLAB as `normcdf(0.67)`.

b) What is the distribution of heights if they are expressed in centimeters? In meters? Well, 1in = 2.54cm. Let Y by a person's height in cemtimeters. Then $Y = aX$, and

$$\begin{aligned} F_Y(y) &= F_X(y/a) \\ f_Y(y) &= \frac{1}{|a|} f_X(y/a). \end{aligned}$$

Given that $X \sim N(\mu, \sigma^2)$,

$$\begin{aligned} f_Y(y) &= \frac{1}{\sqrt{2\pi a^2 \sigma^2}} \exp\left\{-\frac{1}{2}\left(\frac{y/a - \mu}{\sigma}\right)^2\right\} \\ &= \frac{1}{\sqrt{2\pi a^2 \sigma^2}} \exp\left\{-\frac{1}{2}\left(\frac{y - a\mu}{a\sigma}\right)^2\right\} \end{aligned}$$

so $Y \sim N(a\mu, a^2\sigma^2)$. Here, this implies that the distribution of heights in centimeters is $N(177.8, 58.06)$ as opposed to $N(70, 9)$ for inches, and the distribution in meters is $N(1.778, .0058)$.