Stat 310 Homework 3 Key

Chapter 2, problems 61, 67, 69, Chapter 3, problems 8, 12, 18, 19, 20, 34, 37. Due 9/23/99.

2.61. Find the density of cX when X follows a gamma distribution. Show that only λ is affected by such a transformation, which justifies calling λ a scale parameter.

Let $Y = cX$. Then

$$f_Y(y) = \frac{1}{|c|} f_X(y/c).$$

When X is a Gamma(α, λ) random variable, this becomes

$$\begin{aligned} f_Y(y) &= \frac{\lambda^\alpha}{c\Gamma(\alpha)}(y/c)^{\alpha-1}e^{-\lambda y/c} \\ f_Y(y) &= \frac{(\lambda/c)^\alpha}{\Gamma(\alpha)}y^{\alpha-1}e^{-(\lambda/c)y} \end{aligned}$$

so Y has a Gamma$(\alpha, \lambda/c)$ distribution.

2.67. The Weibull cumulative distribution function is

$$F_X(x) = 1 - e^{-(x/\alpha)^\beta}, \quad x \geq 0, \quad \alpha > 0, \quad \beta > 0.$$

Note: there is a typo in my copy of the book! In my copy, at any rate, the exponent is missing a minus sign. Most of you didn't seem to have this problem, so I hope that got corrected at a later printing.

a) Find the density function. Ok, differentiating with respect to x

$$\begin{aligned} f_x(x) &= -e^{-(x/\alpha)^\beta}\frac{\partial}{\partial x}\left(-(x/\alpha)^\beta\right) \\ &= (\beta/\alpha^\beta)x^{\beta-1}e^{-(x/\alpha)^\beta}. \end{aligned}$$

b) Show that if W follows a Weibull distribution, then $X = (W/\alpha)^\beta$ follows an exponential distribution. Let $g(x) = (x/\alpha)^\beta$. Then $g^{-1}(x) = \alpha x^{1/\beta}$. We know that

$$\begin{aligned} f_X(x) &= f_W(g^{-1}(x))\left|\frac{\partial}{\partial x}g^{-1}(x)\right| \\ &= (\beta/\alpha^\beta)[\alpha x^{1/\beta}]^{\beta-1}e^{-x} * (\alpha/\beta)x^{(1-\beta)/\beta} \\ &= \beta^{1-1}\alpha^{-\beta+\beta-1+1}x^{(\beta-1)/\beta+(1-\beta)/\beta}e^{-x} \\ &= e^{-x}, \quad 0 < x < \infty \end{aligned}$$

so X follows an exponential distribution.

c) How could Weibull random variables be generated from a uniform random number generator? Let U be a uniform(0,1) random variable. We showed in class that $-\log(U)$ follows an exponential distribution. Combining this with the result from part b), $Y = \alpha[-\log(U)]^{1/\beta}$ will follow a Weibull(α, β) distribution.

2.69. If the radius of a sphere is an exponential random variable, find the density function of the volume.

Let

$$g(x) = \frac{4\pi}{3}x^3; \quad g^{-1}(x) = \left(\frac{3}{4\pi}x\right)^{1/3}.$$

Letting X denote the radius of the sphere and Y the volume, $Y = g(X)$ so

$$\begin{aligned} f_Y(y) &= f_X(g^{-1}(y))\left|\frac{\partial}{\partial y}g^{-1}(y)\right| \\ &= \lambda \exp\left[\lambda\left(\frac{3}{4\pi}y\right)^{1/3}\right] * \frac{1}{3}\frac{3}{4\pi}\left(\frac{3}{4\pi}y\right)^{-2/3} \\ &= \frac{\lambda}{3}\left(\frac{3}{4\pi}\right)^{1/3} y^{-2/3}\exp\left[\lambda\left(\frac{3}{4\pi}y\right)^{1/3}\right], \qquad 0 \le y < \infty. \end{aligned}$$

3.8. Let X and Y have the joint density

$$f_{XY}(x,y) = \frac{6}{7}(x+y)^2, \quad 0 \le x \le 1, \quad 0 \le y \le 1.$$

a) By integrating over the appropriate regions, find (i) $P(X > Y)$, (ii) $P(X + Y \le 1)$, (iii) $P(X \le 1/2)$.

(i) We can let the limits of integration be 0 to 1 for x, and 0 to x for y, so

$$\begin{aligned} P(X > Y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{x} f_{XY}\,dydx \\ &= \int_0^1\int_0^x \frac{6}{7}(x+y)^2 dydx \\ &= \int_0^1 \left(\frac{6}{7} * \frac{1}{3}(x+y)^3\Big|_0^x\right) dx \\ &= \int_0^1 \frac{2}{7}(8x^3 - x^3)dx \\ &= \int_0^1 2x^3 dx \\ &= \frac{1}{2}x^4\Big|_0^1 = \frac{1}{2}. \end{aligned}$$

For this part, we could have also found this result by symmetry, noting that $P(X > Y) + P(X < Y) = 1$ and realizing that the joint density is symmetric in x and y so that each of the two terms should contribute equally.

(ii) Here, we can again let the range of integration be 0 to 1 for x, but the range for y is now from 0 to $1 - x$. Thus,

$$P(X + Y \le 1) = \int_0^1\int_0^{1-x} \frac{6}{7}(x+y)^2 dydx$$

$$
\begin{aligned}
&= \int_0^1 \left(\frac{6}{7} * \frac{1}{3}(x+y)^3 \Big|_0^{1-x} \right) dx \\
&= \int_0^1 \frac{2}{7}(1-x^3)dx \\
&= \frac{2}{7}\left(x - \frac{1}{4}x^4\right)\Big|_0^1 = \frac{3}{14}.
\end{aligned}
$$

(iii) In this case the limits are easy; from 0 to 1/2 for x, and from 0 to 1 for y.

$$
\begin{aligned}
P(X \leq 1/2) &= \int_0^{1/2} \int_0^1 \frac{6}{7}(x+y)^2 dy dx \\
&= \int_0^{1/2} \frac{2}{7}[(x+1)^3 - x^3] dx \\
&= \frac{2}{7} * \frac{1}{4}[(x+1)^4 - x^4]\Big|_0^{1/2} \\
&= \frac{1}{14}\left[\frac{81}{16} - \frac{1}{16} - 1 + 0\right] = \frac{4}{14}.
\end{aligned}
$$

b) Find the marginal densities of X and Y. This is largely carryover from part a);

$$
\begin{aligned}
f_X &= \int_{-\infty}^{\infty} f_{XY} dy \\
f_X(x) &= \int_0^1 \frac{6}{7}(x+y)^2 dy \\
&= \frac{2}{7}(x+y)^3\Big|_0^1 \\
&= \frac{2}{7}[(x+1)^3 - x^3] \\
&= \frac{2}{7}[3x^2 + 3x + 1], \quad 0 \leq x \leq 1.
\end{aligned}
$$

As the joint density is symmetric in x and y, an identical exercise yields

$$
f_Y(y) = \frac{2}{7}[3y^2 + 3y + 1], \quad 0 \leq y \leq 1.
$$

c) Find the two conditional densities. Well,

$$
\begin{aligned}
f_{X|Y} &= \frac{f_{XY}}{f_Y} \\
f_{X|Y}(x|y) &= \frac{\frac{6}{7}(x+y)^2}{\frac{2}{7}[3y^2 + 3y + 1]} \\
&= 3\frac{(x+y)^2}{[3y^2 + 3y + 1]}, \quad 0 \leq x \leq 1.
\end{aligned}
$$

Arguing by symmetry again gives

$$
f_{Y|X}(y|x) = 3\frac{(x+y)^2}{[3x^2 + 3x + 1]}, \quad 0 \leq y \leq 1.
$$

3.12. Let

$$f_{XY}(x,y) = c(x^2 - y^2)e^{-x}, \quad 0 \le x < \infty, \quad -x \le y \le x.$$

a) Find c. To do this, we need to integrate the joint density over its entire allowable region, set the result equal to 1, and solve for c.

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}\,dydx \\ &= \int_0^{\infty}\int_{-x}^{x} c(x^2-y^2)e^{-x}dydx \\ &= \int_0^{\infty}\left(cx^2y - c\frac{y^3}{3}\right)e^{-x}\Big|_{-x}^{x} dx \\ &= \int_0^{\infty}\left(2cx^3 - 2c\frac{x^3}{3}\right)e^{-x}dx \\ &= \frac{4}{3}c\int_0^{\infty} x^3e^{-x}dx. \end{aligned}$$

Now at this point, we can proceed in two equivalent ways. We can do the integration by parts, getting (after a few splitting into parts)

$$-x^3e^{-x}|_0^{\infty} - 3x^2e^{-x}|_0^{\infty} - 6xe^{-x}|_0^{\infty} + \int_0^{\infty} 6e^{-x}dx = 6,$$

or we can recognize that the above integral is the Gamma function with parameter $3+1=4$, and $\Gamma(4) = (4-1)! = 6$. This integral occurs sufficiently often that it is worth learning to look for it. At any rate,

$$1 = \frac{4}{3}c * 6 = 8c \quad \to \quad c = \frac{1}{8}.$$

b) Find the marginal densities. Well, we went a long way towards finding the marginal density for x in solving part a);

$$\begin{aligned} f_X(x) &= \int_{-x}^{x} c(x^2-y^2)e^{-x}dydx \\ &= \frac{4}{3}cx^3e^{-x} \\ &= \frac{1}{6}x^3e^{-x}, \quad 0 \le x < \infty \end{aligned}$$

which we recognize as a Gamma($\alpha = 4, \lambda = 1$) density. In solving for f_Y, we need to be a bit careful with the limits of integration; for a specific value of y, x ranges from $|y|$ to ∞. It is also easier to split up the x^2 and y^2 parts of the joint density in doing the integration.

$$\begin{aligned} f_Y(y) &= \int_{|y|}^{\infty} cx^2e^{-x}dx - \int_{|y|}^{\infty} cy^2e^{-x}dx \\ &= -cx^2e^{-x}|_{|y|}^{\infty} - 2cxe^{-x}|_{|y|}^{\infty} + \int_{|y|}^{\infty} 2ce^{-x}dx - \int_{|y|}^{\infty} cy^2e^{-x}dx \\ &= c|y|^2e^{-|y|} + 2c|y|e^{-|y|} - 2ce^{-x}|_{|y|}^{\infty} + cy^2e^{-x}|_{|y|}^{\infty} \\ &= c[(y^2+2|y|+2) - y^2]e^{-|y|} \\ &= \frac{1}{4}(|y|+1)e^{-|y|}, \quad -\infty < y < \infty. \end{aligned}$$

c) Find the conditional densities. Ok.

$$\begin{aligned} f_{Y|X}(y|x) &= \frac{f_{XY}(x,y)}{f_X(x)} \\ &= \frac{c(x^2-y^2)e^{-x}}{\frac{4}{3}cx^3e^{-x}} \\ &= \frac{3}{4} * \frac{x^2-y^2}{x^3}, \quad -x \le y \le x \end{aligned}$$

and

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{f_{XY}(x,y)}{f_Y(y)} \\ &= \frac{c(x^2-y^2)e^{-x}}{2c(|y|+1)e^{-|y|}} \\ &= \frac{1}{2} * \frac{(x^2-y^2)e^{-(x-|y|)}}{|y|+1}, \quad |y| \le x \le \infty. \end{aligned}$$

3.18. Let X and Y have the joint density function

$$f_{XY}(x,y) = k(x-y), \quad 0 \le y \le x \le 1$$

and 0 elsewhere.

a) Sketch the region of integration over which the density is positive and use it in determining limits of integration to answer the following questions. This is a bit difficult to do in a typesetting environment - the region is triangular with vertices at $(0,0)$, $(1,0)$, and $(1,1)$.

b) Find k. Again, we need to integrate the density over its entire positive region, set the value to 1, and solve for k.

$$\begin{aligned} 1 &= \int_0^1 \int_0^x k(x-y)dydx \\ &= \int_0^1 k\left(xy - \frac{y^2}{2}\right)\Bigg|_0^x dx \\ &= \int_0^1 k\frac{x^2}{2}dx \\ &= k\frac{x^3}{6}\Bigg|_0^1 = \frac{k}{6} \end{aligned}$$

so $k = 6$.

c) Find the marginal densities of X and Y. Starting with stuff from part b),

$$\begin{aligned} f_X(x) &= \int_0^x k(x-y)dy \\ &= k\frac{x^2}{2} = 3x^2, \quad 0 < x < 1. \end{aligned}$$

$$
\begin{aligned}
f_Y(y) &= \int_y^1 k(x-y)dx \\
&= k\left(\frac{x^2}{2} - xy\right)\Bigg|_y^1 \\
&= k\left(\frac{1}{2} - y\right) - k\left(\frac{y^2}{2} - y^2\right) \\
&= k\left(\frac{y^2}{2} - y + \frac{1}{2}\right) \\
&= 3(1-y)^2, \quad 0 < y < 1.
\end{aligned}
$$

d) Find the conditional densities of Y given X and X given Y.

$$
\begin{aligned}
f_{Y|X}(y|x) &= \frac{f_{XY}(x,y)}{f_X(x)} = \frac{6(x-y)}{3x^2} = 2\frac{x-y}{x^2}, \quad 0 < y < x \\
f_{X|Y}(x|y) &= \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{6(x-y)}{3(1-y)^2} = 2\frac{x-y}{(1-y)^2}, \quad y < x < 1
\end{aligned}
$$

3.19. Suppose that two components have independent exponentially distributed lifetimes, T_1 and T_2, with parameters α and β respectively. Find

a) $P(T_1 > T_2)$. OK. The limits of integration here are 0 to t_1 for t_2, and 0 to ∞ for t_1.

$$
\begin{aligned}
P(T_1 > T_2) &= \int_0^\infty \int_0^{t_1} f_{T_1T_2}(t_1, t_2)dt_2dt_1 \\
&= \int_0^\infty \int_0^{t_1} f_{T_1}(t_1)f_{T_2}(t_2)dt_2dt_1 \quad \text{independence} \\
&= \int_0^\infty f_{T_1}(t_1)\left[\int_0^{t_1} f_{T_2}(t_2)dt_2\right]dt_1 \\
&= \int_0^\infty f_{T_1}\left[1 - e^{-\beta t_1}\right]dt_1 \quad \text{cdf of an exponential} \\
&= \int_0^\infty f_{T_1}(t_1)dt_1 - \int_0^\infty \alpha e^{-(\alpha+\beta)t_1}dt_1 \\
&= 1 - \left[-\frac{\alpha}{\alpha+\beta}e^{-(\alpha+\beta)t_1}\Bigg|_0^\infty\right] \\
&= 1 - \frac{\alpha}{\alpha+\beta} = \frac{\beta}{\alpha+\beta}.
\end{aligned}
$$

Note that this does something sensible - if we let $\alpha = \beta$, so that the two random variables are identically distributed, then the chances are equal that one is greater than the other.

b) $P(T_1 > 2T_2)$. This is much like part a), but the limits of integration for t_2 are now from 0 to $t_1/2$.

$$
P(T_1 > 2T_2) = \int_0^\infty \int_0^{t_1/2} f_{T_1T_2}(t_1, t_2)dt_2dt_1
$$

$$\begin{aligned}
&= \int_0^\infty f_{T_1}\left[1 - e^{-\beta t_1/2}\right] dt_1 \quad \text{cdf of an exponential} \\
&= 1 - \int_0^\infty \alpha e^{-(\alpha+\beta/2)t_1} dt_1 \\
&= 1 - \frac{\alpha}{\alpha+\beta/2} = \frac{\beta/2}{\alpha+\beta/2}.
\end{aligned}$$

Note that this is equivalent to what we got for part a) if we change the value of the parameter of T_2 from β to $\beta/2$. This fits with the result we found earlier (problem 2.61) where we found that if X has a Gamma(α, λ) distribution, then cX has a Gamma$(\alpha, \lambda/c)$ distribution. The exponential distribution is a gamma distribution with $\alpha = 1$, so letting $c = 2$ the result follows.

3.20. If X_1 is uniform on $[0, 1]$, and, conditional on X_1, X_2 is uniform on $[0, X_1]$, find the joint and marginal distributions of X_1 and X_2.

Well, from the statement of the problem, the densities that we are given are f_{X_1} and $f_{X_2|X_1}$. These can easily be used to construct the joint density function, as

$$\begin{aligned}
f_{X_1X_2}(x_1, x_2) &= f_{X_1}(x_1) f_{X_2|X_1}(x_2|x_1) \\
&= 1 * \frac{1}{x_1} = \frac{1}{x_1}, \quad 0 < x_1 < 1, \quad 0 < x_2 < x_1.
\end{aligned}$$

Now, starting with the joint we can integrate to get the marginals:

$$\begin{aligned}
f_{X_1}(x_1) &= 1, \quad 0 < x_1 < 1 \quad \text{(given)} \\
f_{X_2}(x_2) &= \int_{x_2}^1 \frac{1}{x_1} dx_1 \\
&= \log(x_1)|_{x_2}^1 \\
&= -\log(x_2), \quad 0 < x_2 < 1.
\end{aligned}$$

3.34. Let N_1 and N_2 be independent random variables following Poisson distributions with parameters λ_1 and λ_2. Show that the distribution of $N = N_1 + N_2$ is Poisson with parameter $\lambda_1 + \lambda_2$.

$$\begin{aligned}
p_N(n) &= \sum_{n_1=-\infty}^{\infty} p_{N_1N_2}(n_1, n - n_1) \\
&= \sum_{n_1=-\infty}^{\infty} p_{N_1}(n_1) p_{N_2}(n - n_1) \quad \text{independence} \\
&= \sum_{n_1=0}^{n} p_{N_1}(n_1) p_{N_2}(n - n_1) \quad \text{pmfs both positive} \\
&= \sum_{n_1=0}^{n} \frac{\lambda_1^{n_1} e^{-\lambda_1}}{n_1!} \frac{\lambda_2^{n-n_1} e^{-\lambda_2}}{(n-n_1)!}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} \sum_{n_1=0}^{n} \frac{n!}{n_1!(n-n_1)!} \lambda_1^{n_1} \lambda_2^{n-n_1} \\
&= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} \sum_{n_1=0}^{n} \binom{n}{n_1} \lambda_1^{n_1} \lambda_2^{n-n_1} \\
&= \frac{e^{-(\lambda_1+\lambda_2)}}{n!} (\lambda_1+\lambda_2)^n
\end{aligned}
$$

which is the pmf of a Poisson random variable with parameter $\lambda_1 + \lambda_2$.

3.37. Let X and Y by independent standard normal random variables. Find the density of $Z = X + Y$, and show that Z is normally distributed as well. (*Hint*: Use the technique of completing the square to help in evaluating the integral.)

As X and Y are independent,

$$
f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x) dx
$$

and as both x and y range from $-\infty$ to ∞ we leave the integration limits alone.

$$
\begin{aligned}
f_Z(z) &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(z-x)^2\right) dx \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}[x^2 + (z-x)^2]\right) dx \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}[2x^2 - 2xz + z^2]\right) dx \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}[2(x - \frac{1}{2}z)^2 + \frac{1}{2}z^2]\right) dx \\
&= \frac{1}{2\pi} \exp\left(-\frac{1}{4}z^2\right) \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}\left[\frac{x - \frac{1}{2}z}{1/\sqrt{2}}\right]^2\right) dx \\
&= \frac{1}{2\pi} \exp\left(-\frac{1}{2}\left[\frac{z}{\sqrt{2}}\right]^2\right) \sqrt{2\pi(1/\sqrt{2})^2} \\
&= \frac{1}{\sqrt{2\pi(\sqrt{2})^2}} \exp\left(-\frac{1}{2}\left[\frac{z}{\sqrt{2}}\right]^2\right)
\end{aligned}
$$

which is the density function of a $N(0,2)$ random variable.