

Stat 310 Homework 4 Key

Chapter 3, problems 43, 54, 61, 65, Chapter 4, problems 18, 23, 30, 31, 45, 48. Due 9/30/99.

3.43 Consider forming a random rectangle in two ways. Let U_1 , U_2 , and U_3 be independent random variables uniform on $[0, 1]$. One rectangle has sides U_1 and U_2 , and the other is a square with sides U_3 . Find the probability that the area of the square is greater than the area of the other rectangle.

Ok, this problem has multiple parts. First, we find the distribution of the area of the rectangle, and then we find the distribution of the square. As these random variables are independent, we can then examine the joint distribution of the two areas as the product of the individual distributions and integrate over the appropriate area to get the answer. First, the density of the area of the rectangle: let $A_1 = U_1 * U_2$. We want the density of A_1 . As we showed in class (see also problem 41),

$$\begin{aligned}f_{A_1}(a_1) &= \int f_{U_1 U_2} \left(u_1, \frac{a_1}{u_1} \right) \frac{1}{|u_1|} du_1 \\f_{A_1}(a_1) &= \int f_{U_1}(u_1) f_{U_2} \left(\frac{a_1}{u_1} \right) \frac{1}{|u_1|} du_1\end{aligned}$$

As for the limits of integration, f_{U_1} bounds u_1 above at 1, and f_{U_2} bounds u_1 below at a_1 , so we get

$$\begin{aligned}f_{A_1}(a_1) &= \int_{a_1}^1 f_{U_1}(u_1) f_{U_2} \left(\frac{a_1}{u_1} \right) \frac{1}{|u_1|} du_1 \\&= \int_{a_1}^1 1 * 1 * \frac{1}{u_1} du_1 \\&= -\log(a_1), \quad 0 < a_1 < 1.\end{aligned}$$

Now we need the density of the area of the square: let $A_2 = U_3^2$.

$$\begin{aligned}F_{A_2}(a_2) &= P(A_2 < a_2) \\&= P(U_3^2 < a_2) \\&= P(U_3 < \sqrt{a_2}) \\&= F_{U_3}(\sqrt{a_2}), \quad 0 < a_2 < 1 \\&= \sqrt{a_2}. \\f_{A_2} &= \frac{1}{2\sqrt{a_2}}, \quad 0 < a_2 < 1.\end{aligned}$$

We now need to compute $P(A_1 < A_2)$. As A_1 and A_2 are independent (they do not affect one another) we have the joint density. Both A_1 and A_2 are defined over the unit interval, so the region of integration (where $A_2 > A_1$) is the triangle above the main diagonal, with vertices $(0, 0)$, $(1, 1)$, and $(0, 1)$. Thus,

$$\begin{aligned}P(A_1 < A_2) &= \int \int f_{A_1} f_{A_2} da_1 da_2 \quad \text{limits?} \\&= \int_0^1 \int f_{A_1} f_{A_2} da_1 da_2 \quad \text{set the outer limit first}\end{aligned}$$

$$\begin{aligned}
&= \int_0^1 \int_0^{a_2} f_{A_1} f_{A_2} da_1 da_2 \quad \text{now put in the density} \\
&= \int_0^1 \int_0^{a_2} -\log(a_1) \frac{1}{2\sqrt{a_2}} da_1 da_2 \\
&= \int_0^1 \left[\int_0^{a_2} -\log(a_1) da_1 \right] \frac{1}{2\sqrt{a_2}} da_2 \\
&= \int_0^1 [-a_1 \log(a_1) + a_1]_0^{a_2} \frac{1}{2\sqrt{a_2}} da_2 \\
&= \int_0^1 a_2(1 - \log(a_2)) \frac{1}{2\sqrt{a_2}} da_2 \\
&= \int_0^1 \left[\frac{1}{2} a_2^{1/2} - \frac{1}{2} a_2^{1/2} \log(a_2) \right] da_2 \\
&= \frac{2}{6} a_2^{3/2} - \frac{1}{3} a_2^{3/2} \log(a_2) + \frac{2}{9} a_2^{3/2} \Big|_0^1 \\
&= \frac{5}{9}.
\end{aligned}$$

3.54. Find the joint density of $X+Y$ and X/Y , where X and Y are independent exponential random variables with parameter λ . Show that $X+Y$ and X/Y are independent.

First, the sum. Let $S = X + Y$. Then

$$\begin{aligned}
f_S(s) &= \int f_{XY}(x, s-x) dx \\
&= \int f_X(x) f_Y(s-x) dx \quad \text{independence} \\
&= \int_0^s \lambda e^{-\lambda x} \lambda e^{-\lambda(s-x)} dx \\
&= \lambda^2 \int_0^s e^{-\lambda s} dx \\
&= \lambda^2 e^{-\lambda s} \int_0^s dx \\
&= \lambda^2 s e^{-\lambda s}, \quad 0 < s < \infty
\end{aligned}$$

which we recognize as a gamma(2, λ) density. This is a general phenomenon - an exponential is a gamma(1, λ) random variable; if X is gamma(α_1 , λ) and Y is gamma(α_2 , λ), and X and Y are independent, then $X+Y$ will be a gamma($\alpha_1 + \alpha_2$, λ) random variable. Note that the λ values are the same. Things get messy if this is not so.

Now, the quotient. Let $T = X/Y$. As is shown in the text (p.94-5, and mentioned in class),

$$\begin{aligned}
f_T(t) &= \int |y| f_{XY}(yt, y) dy \\
&= \int |y| f_X(yt) f_Y(y) dy \\
&= \int_0^\infty y \lambda e^{-\lambda yt} \lambda e^{-\lambda y} dy
\end{aligned}$$

$$\begin{aligned}
&= \lambda^2 \int_0^\infty y e^{-\lambda(t+1)y} dy \\
&= -\frac{\lambda}{t+1} y e^{-\lambda(t+1)y} \Big|_0^\infty + \int_0^\infty \frac{\lambda}{1+t} e^{-\lambda(t+1)y} dy \\
&= 0 - \frac{1}{(t+1)^2} e^{-\lambda(t+1)y} \Big|_0^\infty \\
&= (1+t)^{-2}, \quad 0 < t < \infty.
\end{aligned}$$

To show independence, we need to find the joint density of S and T in terms of the joint density of X and Y . Here,

$$\begin{aligned}
g_1(X, Y) &= X + Y \\
g_2(X, Y) &= X/Y \\
h_1(S, T) &= S - \frac{S}{T+1} \\
h_2(S, T) &= \frac{S}{T+1}
\end{aligned}$$

To check this, note that

$$\begin{aligned}
h_1(g_1(X, Y), g_2(X, Y)) &= X + Y - \frac{X + Y}{(X/Y) + 1} \\
&= X + Y - Y = X \\
h_2(g_1(X, Y), g_2(X, Y)) &= \frac{X + Y}{(X/Y) + 1} = Y.
\end{aligned}$$

Now, the joint density of S and T in terms of X and Y is given by

$$\begin{aligned}
f_{ST}(s, t) &= f_{XY}(h_1(s, t), h_2(s, t)) \left| \begin{array}{cc} \frac{\partial}{\partial s} h_1(s, t) & \frac{\partial}{\partial t} h_1(s, t) \\ \frac{\partial}{\partial s} h_2(s, t) & \frac{\partial}{\partial t} h_2(s, t) \end{array} \right| \\
&= f_X(h_1(s, t)) f_Y(h_2(s, t)) \left| \begin{array}{cc} 1 - \frac{1}{t+1} & \frac{s}{(t+1)^2} \\ \frac{1}{t+1} & -\frac{s}{(t+1)^2} \end{array} \right| \\
&= \lambda \exp\left(-\lambda * \left[s - \frac{s}{t+1}\right]\right) \lambda \exp\left(-\lambda * \left[\frac{s}{t+1}\right]\right) \left| \frac{-s}{(t+1)^2} \right| \\
&= \lambda^2 s e^{-\lambda s} * \frac{1}{(1+t)^2} \\
&= f_S(s) f_T(t).
\end{aligned}$$

The joint density factors, hence the two quantities are independent. (The ranges of all variables are from 0 to ∞ ; they can be checked at each stage.)

3.61. Let X_1, X_2, \dots, X_n be independent random variables, each with the density function f . Find an expression for the probability that the interval $(-\infty, X_{(n)}]$ encompasses at least 100% of the probability mass of f .

Ok, here we're working with the largest order statistic. The cdf of the largest order statistic is

$$F_{X_{(n)}}(x_{(n)}) = [F(x_{(n)})]^n.$$

Now, for the probability to be at least 100%, we must have $x_{(n)} \geq c$, where $F(c) = \nu$. Hence, the value of c is given by $c = F^{-1}(\nu)$. The chance of being less than or equal to c is

$$\begin{aligned} F_{X_{(n)}}(c) &= [F(F^{-1}(\nu))]^n \\ &= \nu^n, \quad 0 < \nu < 1, \end{aligned}$$

and the probability that we want is

$$1 - F_{X_{(n)}}(c) = 1 - \nu^n, \quad 0 < \nu < 1.$$

3.65. Use the differential method to find the joint density of $X_{(i)}$ and $X_{(j)}$, where $i < j$.

The “differential method” is a fancy way of saying “think carefully about what’s going on.” For the joint density of $X_{(i)}$ and $X_{(j)}$, let us denote the values they take on by u and v . For $f_{i,j}(u, v)$ we note that this implies that one value falls in the interval $(u, u + du)$, one falls in the interval $(v, v + dv)$, $i - 1$ are less than u , $n - j$ are greater than $v + dv$, and $j - i - 1$ are between $u + du$ and v . The probability of any such arrangement is

$$F(u)^{i-1} f(u) [F(v) - F(u)]^{j-i-1} f(v) [1 - F(v)]^{n-j}$$

and the number of such arrangements is given by the multinomial theorem as

$$\binom{n}{(i-1) \ 1 \ (j-i-1) \ 1 \ (n-j)}$$

so the joint density is

$$\begin{aligned} f_{i,j}(u, v) &= \frac{n!}{(i-1)!(j-i-1)!(n-j)!} * \\ &F(u)^{i-1} f(u) [F(v) - F(u)]^{j-i-1} f(v) [1 - F(v)]^{n-j}, \quad X_{(i)} \leq X_{(j)}. \end{aligned}$$

4.18. If U_1, \dots, U_n are independent uniform random variables, find $E(U_{(n)} - U_{(1)})$.

Having just found the joint density of two order statistics, let’s use it.

$$\begin{aligned} E(U_{(n)} - U_{(1)}) &= \int \int (v - u) f_{1,n}(u, v) du dv \\ &= \int_0^1 \int_0^v (v - u) \frac{n!}{(n-2)!} 1 * 1 * (v - u)^{n-2} * 1 * 1 du dv \\ &= \int_0^1 \int_0^v n(n-1)(v - u)^{n-1} du dv \\ &= \int_0^1 [-(n-1)(v - u)^n]_0^v dv \\ &= \int_0^1 (n-1)v^n dv \\ &= \frac{n-1}{n+1}. \end{aligned}$$

Of course, we could have found this another way. In particular, we could have used the fact that the expectation is a linear operator to write

$$\begin{aligned}
 E(U_{(n)} - U_{(1)}) &= E(U_{(n)}) - E(U_{(1)}) \\
 &= \int_0^1 v * f_n dv - \int_0^1 u * f_1 du \\
 &= \int_0^1 n v^n dv - \int_0^1 n u (1 - u)^{n-1} du \\
 &= \frac{n}{n+1} - \int_0^1 n [-(1-u)^n + (1-u)^{n-1}] du \\
 &= \frac{n}{n+1} - \left[\frac{n}{n+1} (1-u)^{n+1} - (1-u)^n \right]_0^1 \\
 &= \frac{n}{n+1} - \frac{1}{n+1}.
 \end{aligned}$$

In the end, we get the same thing, but here we see what the expected values of the individual order statistics are as well.

4.23 Repeat problems 21 and 22 assuming that the distribution of the lengths is exponential.

Ok, so it's cheating a bit. Let's look at the equivalent of problem 21, specifically, a random square has a side length that is an exponential random variable. Find the expected area of the square. If we let X denote the side length, what we're looking for is $E(X^2)$. This is

$$\begin{aligned}
 E(X^2) &= \int x^2 f_X(x) dx \\
 &= \int_0^\infty x^2 \lambda e^{-\lambda x} dx \\
 &= -x^2 e^{-\lambda x} \Big|_0^\infty + \int_0^\infty 2x e^{-\lambda x} dx \\
 &= 0 + -\frac{2x}{\lambda} e^{-\lambda x} \Big|_0^\infty + \int_0^\infty \frac{2}{\lambda} e^{-\lambda x} dx \\
 &= 0 + -\frac{2}{\lambda^2} e^{-\lambda x} \Big|_0^\infty \\
 &= \frac{2}{\lambda^2}
 \end{aligned}$$

Now let's look at the equivalent of problem 22, namely that a random rectangle has sides the lengths of which are independent exponential random variables. Find the expected area of the rectangle, and compare this result to that found for the square above. If we let Y_1 and Y_2 denote the lengths of the sides, then the area we are looking for is $E(Y_1 Y_2) = E(Y_1) E(Y_2)$, where the factorization follows from the independence of Y_1 and Y_2 . As Y_1 and Y_2 have the same distribution, we just need to find $E(Y_1)$.

$$\begin{aligned}
 E(Y_1) &= \int y_1 f_{Y_1}(y_1) dy_1 \\
 &= \int_0^\infty y_1 \lambda e^{-\lambda y_1} dy_1
 \end{aligned}$$

$$\begin{aligned}
&= -y_1 e^{-\lambda y_1} \Big|_0^\infty + \int_0^\infty \frac{1}{\lambda} e^{-\lambda y_1} dy_1 \\
&= 0 + -\frac{1}{\lambda} e^{-\lambda y_1} \Big|_0^\infty \\
&= \frac{1}{\lambda}
\end{aligned}$$

so that the expected area of the rectangle is $1/\lambda^2$, which is smaller than the expected area of the square.

4.30. Find $E[1/(X + 1)]$, where X is a Poisson random variable.

$$\begin{aligned}
E[1/(X + 1)] &= \sum_{i=0}^{\infty} \frac{1}{i + 1} \frac{\lambda^i e^{-\lambda}}{i!} \\
&= \frac{1}{\lambda} \sum_{i=0}^{\infty} \frac{\lambda^{i+1} e^{-\lambda}}{(i + 1)!} \\
&= \frac{1}{\lambda} \left[\left(\sum_{j=0}^{\infty} \frac{\lambda^j e^{-\lambda}}{j!} \right) - \frac{\lambda^0 e^{-\lambda}}{0!} \right] \\
&= \frac{1}{\lambda} [1 - e^{-\lambda}].
\end{aligned}$$

4.31. Let X be uniformly distributed in the interval $[1, 2]$. Find $E(1/X)$. Is $E(1/X) = 1/E(X)$?

Ok, to begin with,

$$\begin{aligned}
E(1/X) &= \int \frac{1}{x} f_X(x) dx \\
&= \int_1^2 \frac{1}{x} dx \\
&= \log(2) - \log(1) = \log(2) = 0.6931.
\end{aligned}$$

As for the other function,

$$\begin{aligned}
E(X) &= \int x f_X(x) dx \\
&= \int_1^2 x dx \\
&= \frac{x^2}{2} \Big|_1^2 = \frac{3}{2}, \\
1/E(X) &= 2/3
\end{aligned}$$

which is not the same as what we found above! In general, we cannot assume that $E(g(X)) = g(E(X))$. In passing, we note that the expected value of the random variable is right where we would have guessed it was by symmetry, in the middle of the interval.

4.45. Two independent measurements, X and Y are taken of a quantity μ . $E(X) = E(Y) = \mu$, but σ_X and σ_Y are unequal. The two measurements are combined by means of a weighted average to give

$$Z = \alpha X + (1 - \alpha)Y$$

where α is a scalar and $0 \leq \alpha \leq 1$.

a) Show that $E(Z) = \mu$. This simply uses the linearity of the expectation operator, in that

$$\begin{aligned} E(Z) &= E(\alpha X + (1 - \alpha)Y) \\ &= \alpha E(X) + (1 - \alpha)E(Y) \\ &= \alpha\mu + (1 - \alpha)\mu = \mu. \end{aligned}$$

b) Find α in terms of σ_X and σ_Y to minimize $Var(Z)$. Ok, to do this we need an expression for $Var(Z)$.

$$\begin{aligned} Var(Z) &= Var(\alpha X + (1 - \alpha)Y) \\ &= Var(\alpha X) + Var((1 - \alpha)Y) \quad \text{independence} \\ &= \alpha^2 Var(X) + (1 - \alpha)^2 Var(Y) \quad \text{Thm.A, 4.2} \\ &= \alpha^2 \sigma_X^2 + (1 - \alpha)^2 \sigma_Y^2. \end{aligned}$$

To find the optimal value of α , we differentiate the above equation, treating the values of σ_X and σ_Y as constants, and set it equal to zero. This gives

$$\begin{aligned} 2\alpha\sigma_X^2 &= 2(1 - \alpha)\sigma_Y^2 \\ \alpha(\sigma_X^2 + \sigma_Y^2) &= \sigma_Y^2 \\ \alpha &= \frac{\sigma_Y^2}{\sigma_X^2 + \sigma_Y^2}. \end{aligned}$$

c) Under what circumstances is it better to use the average $(X + Y)/2$ than either X or Y alone? Well, this is simply a special case of the situation considered above, with $\alpha = 1/2$. Averaging does not affect the location of the mean (the target value) so considerations of better rest solely on variance considerations. With $\alpha = 1/2$,

$$Var(Z) = \frac{1}{4}(\sigma_X^2 + \sigma_Y^2).$$

This will exceed the variance of X if $\sigma_Y^2 \geq 3\sigma_X^2$, and it will exceed the variance of Y if $\sigma_X^2 \geq 3\sigma_Y^2$, so simple averaging improves things if

$$\frac{1}{3}\sigma_Y^2 \leq \sigma_X^2 \leq 3\sigma_Y^2.$$

If one machine gives much more variable results than the other, then the variability of average will be dominated by this, making it worse than just using the single measurement from the more precise machine. Note, that we can still combine the results and get a better estimate than we started with - we just have to weight the result from the more precise machine more heavily.

4.48. Let X , Y and Z be uncorrelated random variables with variances σ_X^2 , σ_Y^2 , and σ_Z^2 , respectively. Let

$$\begin{aligned}U &= Z + X \\V &= Z + Y\end{aligned}$$

Find $Cov(U, V)$ and ρ_{UV} .

Start with the covariance.

$$\begin{aligned}Cov(U, V) &= E(UV) - E(U)E(V) \quad \text{by definition} \\&= E((Z + X)(Z + Y)) - E(Z + X)E(Z + Y) \\&= E(Z^2 + XZ + XY + YZ) - \\&\quad [E(Z)^2 + E(X)E(Z) + E(X)E(Y) + E(Y)E(Z)] \\&= [E(Z^2) - E(Z)^2] + [E(XZ) - E(X)E(Z)] + \\&\quad [E(YZ) - E(Y)E(Z)] + [E(XY) - E(X)E(Y)] \\&= Var(Z) + Cov(X, Z) + Cov(Y, Z) + Cov(X, Y) \\&= Var(Z) = \sigma_Z^2. \quad \text{others uncorrelated}\end{aligned}$$

Now for the correlation:

$$\begin{aligned}\rho_{UV} &= \frac{Cov(U, V)}{\sqrt{Var(U)Var(V)}} \quad \text{by definition} \\Var(U) &= Var(Z + X) \\&= Var(Z) + Var(X) = \sigma_Z^2 + \sigma_X^2 \quad \text{uncorrelated} \\Var(V) &= \sigma_Z^2 + \sigma_Y^2 \quad \text{same reasoning} \\ \rho_{UV} &= \frac{\sigma_Z^2}{\sqrt{(\sigma_Z^2 + \sigma_X^2)(\sigma_Z^2 + \sigma_Y^2)}}\end{aligned}$$

Does this make sense? If the variance of Z is much larger than the variance of the other two, then the changes in the values of U and V will be dominated by the changes in Z and the correlation will be near 1. That makes sense. Similarly, if the variance of Z is much less than the variance of the other two, then the changes in U and V will be only slightly affected by the changes in Z and the correlation will be near zero. That makes sense too.