Stat 310 Homework 7 Key

Chapter 8, problems 14, 15, 16, 19, 39, 42, 44, 45, 57, 58. Due 11/11/99.

8.14 Consider an i.i.d. sample of random variables with density function

$$f(x|\sigma) = \frac{1}{2\sigma} \exp\left(-\frac{|x|}{\sigma}\right)$$

a) Find the method of moments estimate of σ. To do this, we need to find the first few moments of X. The density of X is symmetric about 0, so $E(X) = 0$, which means we can't use this moment to estimate σ. Moving on to the next moment,

$$\begin{aligned}
E(X^2) &= \int_{-\infty}^{\infty} \frac{x^2}{2\sigma} \exp\left(-\frac{|x|}{\sigma}\right) dx \\
&= \int_{-\infty}^{0} \frac{x^2}{2\sigma} \exp\left(\frac{x}{\sigma}\right) dx + \int_{0}^{\infty} \frac{x^2}{2\sigma} \exp\left(-\frac{x}{\sigma}\right) dx \\
&= \frac{x^2}{2} \exp\left(\frac{x}{\sigma}\right)\Big|_{-\infty}^{0} - \int_{-\infty}^{0} x \exp\left(\frac{x}{\sigma}\right) dx - \frac{x^2}{2} \exp\left(-\frac{x}{\sigma}\right)\Big|_{0}^{\infty} + \int_{0}^{\infty} x \exp\left(-\frac{x}{\sigma}\right) dx \\
&= -\sigma x \exp\left(\frac{x}{\sigma}\right)\Big|_{-\infty}^{0} + \int_{-\infty}^{0} \sigma \exp\left(\frac{x}{\sigma}\right) dx + \sigma x \exp\left(-\frac{x}{\sigma}\right)\Big|_{0}^{\infty} - \int_{0}^{\infty} \sigma \exp\left(-\frac{x}{\sigma}\right) dx \\
&= \sigma^2 \exp\left(\frac{x}{\sigma}\right)\Big|_{-\infty}^{0} + -\sigma^2 \exp\left(-\frac{x}{\sigma}\right)\Big|_{0}^{\infty} = 2\sigma^2.
\end{aligned}$$

Thus, substituting the sample second moment for the true second moment gives

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^{n} x_i^2 &= 2\hat{\sigma}^2 \\
\hat{\sigma} &= \sqrt{\frac{1}{2n}\sum_{i=1}^{n} x_i^2}.
\end{aligned}$$

b) Find the maximum likelihood estimate of σ. Well, we have the density function of the x_i's, so we find the likelihood function, take logs, and differentiate.

$$\begin{aligned}
L(\sigma) &= \prod_{i=1}^{n} \frac{1}{2\sigma} \exp\left(-\frac{|x_i|}{\sigma}\right) \\
l(\sigma) &= \sum_{i=1}^{n} \left[-\log(2) - \log(\sigma) - \frac{|x_i|}{2\sigma}\right] \\
l'(\sigma) &= \sum_{i=1}^{n} \left[-\frac{1}{\sigma} + \frac{|x_i|}{\sigma^2}\right] \\
&= -\frac{n}{\sigma} + \frac{\sum_{i=1}^{n} |x_i|}{\sigma^2}.
\end{aligned}$$

Setting this last expression equal to zero gives

$$\hat{\sigma}_{ML} = \frac{1}{n}\sum_{i=1}^{n} |x_i|.$$

c) Find the asymptotic variance of the mle. Well, the asymptotic variance of the mle is given by $V(\hat{\sigma}) = -1/l''(\hat{\sigma})$. The second derivative of the log-likelihood function is

$$\begin{aligned} l''(\sigma) &= \frac{n}{\sigma^2} - \frac{2\sum_{i=1}^n |x_i|}{\sigma^3}, \\ l''(\hat{\sigma}) &= \frac{n}{\hat{\sigma}^2} - \frac{2n}{\hat{\sigma}^2} = -\frac{n}{\hat{\sigma}^2} \\ V(\hat{\sigma}) &= \frac{\hat{\sigma}^2}{n}. \end{aligned}$$

d) Find a sufficient statistic for σ. Well, by inspection the log-likelihood function only involves the x_i's through the sum of the absolute values of the observations. Hence, $\sum_{i=1}^n |x_i|$ is a sufficient statistic for σ.

8.15 Suppose that $X_1, \ldots, X_n$ are i.i.d. random variables on the interval $[0, 1]$ with the density function

$$f(x|a) = \frac{\Gamma(2\alpha)}{\Gamma(\alpha)^2}[x(1-x)]^{\alpha-1}$$

where $\alpha > 0$ is a parameter to be estimated from the sample. It can be shown that

$$\begin{aligned} E(X) &= \frac{1}{2}, \\ V(X) &= \frac{1}{4(2\alpha+1)}. \end{aligned}$$

a) How does the shape of the density depend on α? Well, the density is symmetric about $x = 1/2$ for all values of α; increasing α concentrates the density ever more tightly about 1/2.

b) How can the method of moments be used to estimate α? Well, the first moment does not involve α explicitly so we must work with the second moment.

$$\begin{aligned} E(X^2) &= V(X) + E(X)^2 \\ &= \frac{1}{4(2\alpha+1)} + \frac{1}{4} \\ 4E(X^2) - 1 &= \frac{1}{2\alpha+1} \\ (2(4E(X^2)-1))^{-1} &= \alpha + 1/2 \\ \alpha &= (2(4E(X^2)-1))^{-1} - 1/2 \\ \hat{\alpha} &= \left[2\left(4\left[\frac{1}{n}\sum_{i=1}^n X_i^2\right] - 1\right)\right]^{-1} - 1/2 \end{aligned}$$

c) What equation does the maximum likelihood estimate of α satisfy?

$$L(\alpha) = \prod_{i=1}^n \frac{\Gamma(2\alpha)}{\Gamma(\alpha)^2}[x_i(1-x_i)]^{\alpha-1}$$

$$
\begin{aligned}
l(\alpha) &= \sum_{i=1}^{n} [\log(\Gamma(2\alpha)) - 2\log(\Gamma(\alpha)) + (\alpha - 1)\log(x_i(1-x_i))] \\
l'(\alpha) &= \sum_{i=1}^{n} \left[\frac{2\Gamma'(2\alpha)}{\Gamma(2\alpha)} - 2\frac{\Gamma'(\alpha)}{\Gamma(\alpha)} + \log(x_i(1-x_i))\right] \\
&= 2n\frac{\Gamma'(2\alpha)}{\Gamma(2\alpha)} - 2n\frac{\Gamma'(\alpha)}{\Gamma(\alpha)} + \sum_{i=1}^{n} \log(x_i(1-x_i))
\end{aligned}
$$

Setting this last expression equal to zero, we see that the mle must satisfy the equation

$$
\frac{\Gamma'(2\alpha)}{\Gamma(2\alpha)} - \frac{\Gamma'(\alpha)}{\Gamma(\alpha)} = \frac{1}{2n}\sum_{i=1}^{n} \log(x_i(1-x_i)).
$$

d) What is the asymptotic variance of the maximum likelihood estimate? To find the asymptotic variance, we need the second derivative of the log-likelihood function,

$$
\begin{aligned}
l''(\alpha) &= 2n\frac{2\Gamma''(2\alpha)}{\Gamma(2\alpha)} - 2n\frac{2\Gamma'(2\alpha)^2}{\Gamma(2\alpha)^2} + 2n\frac{\Gamma''(\alpha)}{\Gamma(\alpha)} - 2n\frac{\Gamma'(\alpha)^2}{\Gamma(\alpha)^2}, \\
V(\hat{\alpha}) &= -\frac{1}{2n}\left[\frac{2\Gamma''(2\hat{\alpha})}{\Gamma(2\hat{\alpha})} - \frac{2\Gamma'(2\hat{\alpha})^2}{\Gamma(2\hat{\alpha})^2} + \frac{\Gamma''(\hat{\alpha})}{\Gamma(\hat{\alpha})} - \frac{\Gamma'(\hat{\alpha})^2}{\Gamma(\hat{\alpha})^2},\right]^{-1}
\end{aligned}
$$

e) Find a sufficient statistic for α. By inspection, the log-likelihood function only involves the data values through $\sum \log(x_i(1-x_i))$, so this (or, equivalently, $\prod(x_i(1-x_i))$) is a sufficient statistic for α.

8.16 Suppose that $X_1, \ldots, X_n$ are i.i.d. random variables on the interval $[0, 1]$ with the density function

$$
f(x|a) = \frac{\Gamma(3\alpha)}{\Gamma(\alpha)\Gamma(2\alpha)} x^{\alpha-1}(1-x)^{2\alpha-1}
$$

where $\alpha > 0$ is a parameter to be estimated from the sample. It can be shown that

$$
\begin{aligned}
E(X) &= \frac{1}{3}, \\
V(X) &= \frac{2}{9(3\alpha+1)}.
\end{aligned}
$$

a) How could the method of moments be used to estimate α? Well, the first moment does not involve α explicitly so we must work with the second moment.

$$
\begin{aligned}
E(X^2) &= V(X) + E(X)^2 \\
&= \frac{2}{9(3\alpha+1)} + \frac{1}{9} \\
9E(X^2) - 1 &= \frac{2}{3\alpha+1} \\
2(3(9E(X^2)-1))^{-1} &= \alpha + 1/3 \\
\alpha &= 2(3(9E(X^2)-1))^{-1} - 1/3 \\
\hat{\alpha} &= 2\left[3\left(9\left[\frac{1}{n}\sum_{i=1}^{n} X_i^2\right] - 1\right)\right]^{-1} - 1/3.
\end{aligned}
$$

c) What equation does the maximum likelihood estimate of α satisfy?

$$\begin{aligned}
L(\alpha) &= \prod_{i=1}^{n} \frac{\Gamma(3\alpha)}{\Gamma(\alpha)\Gamma(2\alpha)} x_i^{\alpha-1}(1-x_i)^{2\alpha-1} \\
l(\alpha) &= \sum_{i=1}^{n} \left[\log(\Gamma(3\alpha)) - \log(\Gamma(\alpha)) - \log(\Gamma(2\alpha)) + (\alpha-1)\log(x_i) + (2\alpha-1)\log(1-x_i)\right] \\
l'(\alpha) &= \sum_{i=1}^{n} \left[\frac{3\Gamma'(3\alpha)}{\Gamma(3\alpha)} - \frac{\Gamma'(\alpha)}{\Gamma(\alpha)} - \frac{2\Gamma'(2\alpha)}{\Gamma(2\alpha)} + \log(x_i) + 2\log(1-x_i)\right] \\
&= n\frac{3\Gamma'(3\alpha)}{\Gamma(3\alpha)} - n\frac{\Gamma'(\alpha)}{\Gamma(\alpha)} - n\frac{2\Gamma'(2\alpha)}{\Gamma(2\alpha)} + \sum_{i=1}^{n}\left[\log(x_i) + 2\log(1-x_i)\right]
\end{aligned}$$

Setting this last expression equal to zero, we see that the mle must satisfy the equation

$$\frac{3\Gamma'(3\alpha)}{\Gamma(3\alpha)} - \frac{\Gamma'(\alpha)}{\Gamma(\alpha)} - \frac{2\Gamma'(2\alpha)}{\Gamma(2\alpha)} = \frac{1}{n}\sum_{i=1}^{n}\left[\log(x_i) + 2\log(1-x_i)\right].$$

d) What is the asymptotic variance of the maximum likelihood estimate? To find the asymptotic variance, we need the second derivative of the log-likelihood function,

$$\begin{aligned}
l''(\alpha) &= n\frac{9\Gamma''(3\alpha)}{\Gamma(3\alpha)} - n\frac{9\Gamma'(3\alpha)^2}{\Gamma(3\alpha)^2} - n\frac{\Gamma''(\alpha)}{\Gamma(\alpha)} + n\frac{\Gamma'(\alpha)^2}{\Gamma(\alpha)^2} - n\frac{4\Gamma''(2\alpha)}{\Gamma(2\alpha)} + n\frac{4\Gamma'(2\alpha)^2}{\Gamma(2\alpha)^2} \\
V(\hat{\alpha}) &= -\frac{1}{n}\left[\frac{9\Gamma''(3\hat{\alpha})}{\Gamma(3\hat{\alpha})} - \frac{9\Gamma'(3\hat{\alpha})^2}{\Gamma(3\hat{\alpha})^2} - \frac{\Gamma''(\hat{\alpha})}{\Gamma(\hat{\alpha})} + \frac{\Gamma'(\hat{\alpha})^2}{\Gamma(\hat{\alpha})^2} - \frac{4\Gamma''(2\hat{\alpha})}{\Gamma(2\hat{\alpha})} + \frac{4\Gamma'(2\hat{\alpha})^2}{\Gamma(2\hat{\alpha})^2}\right]^{-1}
\end{aligned}$$

e) Find a sufficient statistic for α. By inspection, the log-likelihood function only involves the data values through $\sum[\log(x_i) + 2\log(1-x_i)]$, so this (or, equivalently, $\sum[\log(x_i(1-x_i)^2)]$ or $\prod(x_i(1-x_i)^2)$) is a sufficient statistic for α.

8.19 Suppose that $X_1, \ldots, X_n$ are i.i.d. with density function

$$f(x|\theta) = e^{-(x-\theta)}, \quad x \geq \theta$$

and $f(x|\theta) = 0$ otherwise.

a) Find the method of moments estimate of θ.

$$\begin{aligned}
E(X) &= \int_{\theta}^{\infty} x e^{-(x-\theta)} dx \\
&= \int_{\theta}^{\infty} (x-\theta)e^{-(x-\theta)} dx + \int_{\theta}^{\infty} \theta e^{-(x-\theta)} dx \\
&= 1 + \theta
\end{aligned}$$

where to evaluate the first integral we have made use of the fact that X has a shifted exponential with decay parameter $\lambda = 1$. The method of moments estimator is thus

$$\hat{\theta} = \bar{x} - 1.$$

b) Find the mle of θ. (*Hint:* Be careful, don't differentiate before thinking. For what values of θ is the likelihood positive?)

$$\begin{aligned} L(\theta) &= \prod_{i=1}^{n} e^{-(x_i-\theta)}, \\ l(\theta) &= \sum_{i=1}^{n}[-x_i+\theta] \\ &= n\theta - \sum x_i. \end{aligned}$$

In this setting, we can increase the likelihood by making θ as large as possible. The constraining factor is one that does not show up explicitly in the likelihood, namely that the contribution to the likelihood is a factor of 0, not $e^{-(x_i-\theta)}$, if $x_i < \theta$. Thus, for the likelihood to be positive, all of the x_i must be greater than or equal to θ. As we are trying to make θ as large as possible, we can start low and increase θ until we hit the smallest of the x_i's. This smallest observed value is our maximum likelihood estimate for θ;

$$\hat{\theta} = x_{(1)}.$$

c) Find a sufficient statistic for θ. In this case, once we know the smallest of the observed values, knowing the remainder of the values tells us nothing more with respect to estimating θ. Thus, $x_{(1)}$ is a sufficient statistic for θ.

8.39 The Pareto distribution has been usedin economics as a model for a density function with a slowly decaying tail:

$$f(x|x_0,\theta) = \theta x_0^{\theta} x^{-(\theta+1)}, \quad x \geq x_0, \quad \theta > 1.$$

Assume that $x_0 > 0$ is given and that $X_1, \ldots, X_n$ is an i.i.d. sample.

a) Find the method of moments estimate of θ.

$$\begin{aligned} E(X) &= \int_{x_0}^{\infty} \theta x_0^{\theta} x^{-\theta} dx \\ &= \theta x_0^{\theta} \int_{x_0}^{\infty} x^{-\theta} dx \\ &= \theta x_0^{\theta} \left. \frac{-x^{-(\theta-1)}}{\theta-1} \right|_{x_0}^{\infty} \\ &= \frac{\theta x_0}{\theta - 1} \\ \theta E(X) - E(X) &= \theta x_0 \\ \theta(E(X) - x_0) &= E(X) \\ \theta &= \frac{E(X)}{E(X) - x_0} \\ \hat{\theta} &= \frac{\bar{X}}{\bar{X} - x_0}. \end{aligned}$$

b) Find the mle of θ.

$$\begin{aligned} L(\theta) &= \prod_{i=1}^{n} \theta x_0^{\theta} x_i^{-(\theta+1)} \\ l(\theta) &= \sum_{i=1}^{n} [\log(\theta) + \theta \log(x_0) - (\theta+1)\log(x_i)] \\ l'(\theta) &= \sum_{i=1}^{n} \left[\frac{1}{\theta} + \log(x_0) - \log(x_i)\right] \\ &= \frac{n}{\theta} - \sum_{i=1}^{n} \log(x_i/x_0), \\ \hat{\theta} &= \left(\frac{1}{n}\sum_{i=1}^{n} \log(x_i/x_0)\right)^{-1}. \end{aligned}$$

c) Find the asymptotic variance of the mle.

$$\begin{aligned} l''(\theta) &= -\frac{n}{\theta^2} \\ V(\hat{\theta}) &= \frac{\hat{\theta}^2}{n}. \end{aligned}$$

d) Find a sufficient statistic for θ. By inspection of the log-likelihood function, $\sum \log(x_i)$ is a sufficient statistic for θ, given that x_0 is known.

8.42 Let $X_1, \ldots, X_n$ be an i.i.d. sample from a Rayleigh distribution with parameter $\theta > 0$:

$$f(x|\theta) = \frac{x}{\theta^2} e^{-x^2/2\theta^2}, \quad x \geq 0.$$

a) Find the method of moments estimate of θ.

$$\begin{aligned} E(X) &= \int_0^{\infty} \frac{x^2}{\theta^2} e^{-x^2/2\theta^2}\,dx \\ &= -xe^{-x^2/2\theta^2}\Big|_0^{\infty} + \int_0^{\infty} e^{-x^2/2\theta^2}\,dx \\ &= \frac{1}{2}\int_{-\infty}^{\infty} e^{-x^2/2\theta^2}\,dx \\ &= \frac{1}{2}\sqrt{2\pi\theta^2}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\theta^2}} e^{-x^2/2\theta^2}\,dx \\ &= \frac{1}{2}\sqrt{2\pi\theta^2} = \sqrt{\frac{\pi}{2}}\theta. \\ \hat{\theta} &= \sqrt{\frac{2}{\pi}}\bar{X}. \end{aligned}$$

b) Find the mle of θ.

$$L(\theta) = \prod_{i=1}^{n} \frac{x_i}{\theta^2} e^{-x_i^2/2\theta^2}$$

$$
\begin{aligned}
l(\theta) &= \sum_{i=1}^{n}\left[\log(x_i) - 2\log(\theta) - \frac{x_i^2}{2\theta^2}\right] \\
l'(\theta) &= \sum_{i=1}^{n}\left[-\frac{2}{\theta} + \frac{x_i^2}{\theta^3}\right] \\
&= -\frac{2n}{\theta} + \frac{\sum x_i^2}{\theta^3} \\
2n\hat{\theta}^2 &= \sum_{i=1}^{n} x_i^2 \\
\hat{\theta} &= \sqrt{\frac{\sum x_i^2}{2n}}
\end{aligned}
$$

c) Find the asymptotic variance of the mle.

$$
\begin{aligned}
l''(\theta) &= \frac{2n}{\theta^2} - \frac{3\sum x_i^2}{\theta^4} \\
l''(\hat{\theta}) &= \frac{2n}{\hat{\theta}^2} - \frac{6n}{\hat{\theta}^2} = -\frac{4n}{\hat{\theta}^2} \\
V(\hat{\theta}) &= \frac{\hat{\theta}}{4n}.
\end{aligned}
$$

8.44 Let $X_1, \ldots, X_n$ be i.i.d. random variables with the density function

$$f(x|\theta) = (\theta+1)x^\theta, \quad 0 \leq x \leq 1.$$

a) Find the method of moments estimate of θ.

$$
\begin{aligned}
E(X) &= \int_0^1 (\theta+1)x^{\theta+1}dx \\
&= \frac{\theta+1}{\theta+2} \\
(\theta+2)E(X) &= \theta+1 \\
\theta(E(X)-1) &= 1-2E(X) \\
\theta &= \frac{1-2E(X)}{E(X)-1} \\
\hat{\theta} &= \frac{1-2\bar{X}}{\bar{X}-1} = \frac{1}{1-\bar{X}} - 2.
\end{aligned}
$$

b) Find the mle of θ.

$$
\begin{aligned}
L(\theta) &= \prod_{i=1}^{n}(\theta+1)x_i^\theta \\
l(\theta) &= \sum_{i=1}^{n}[\log(\theta+1) + \theta\log(x_i)]
\end{aligned}
$$

$$\begin{aligned}
l'(\theta) &= \sum_{i=1}^{n}\left[\frac{1}{\theta+1}+\log(x_i)\right] \\
&= \frac{n}{\theta+1}+\sum_{i=1}^{n}\log(x_i). \\
\frac{1}{\hat{\theta}+1} &= -\frac{1}{n}\sum_{i=1}^{n}\log(x_i) \\
\hat{\theta} &= -\left[\frac{1}{n}\sum_{i=1}^{n}\log(x_i)\right]^{-1}-1.
\end{aligned}$$

c) Find the asymptotic variance of the mle.

$$\begin{aligned}
l''(\theta) &= -\frac{n}{(\theta+1)^2} \\
V(\hat{\theta}) &= \frac{(\hat{\theta}+1)^2}{n}
\end{aligned}$$

d) Find a sufficient statistic for θ. By inspection of the log-likelihood function, $\sum \log(x_i)$ (or, equivalently, $\prod x_i$) is a sufficient statistic for θ.

8.45 Let $X_1, \ldots, X_n$ be i.i.d. uniform on $[0, \theta]$.

a) Find the method of moments estimator of θ and its mean and variance.

$$\begin{aligned}
E(X) &= \int_0^{\theta}\frac{x}{\theta}dx = \frac{\theta}{2} \\
\hat{\theta} &= 2\bar{X}. \\
E(\hat{\theta}) &= E(2\bar{X}) = \frac{2}{n}\sum_{i=1}^{n}E(X_i) = \frac{2}{n}\sum\frac{\theta}{2} = \theta \\
V(\hat{\theta}) &= V(2\bar{X}) = \frac{4}{n^2}\sum_{i=1}^{n}V(X_i) \\
E(X^2) &= \int_0^{\theta}\frac{x^2}{\theta}dx = \frac{\theta^2}{3} \\
V(X) &= \frac{\theta^2}{3}-\frac{\theta^2}{4} = \frac{\theta^2}{12} \\
V(\hat{\theta}) &= \frac{4}{n^2}\sum_{i=1}^{n}\frac{\theta^2}{12} = \frac{\theta^2}{3n}
\end{aligned}$$

b) Find the mle of θ.

$$\begin{aligned}
L(\theta) &= \prod_{i=1}^{n}\frac{1}{\theta} \\
l(\theta) &= -\sum_{i=1}^{n}\log(\theta).
\end{aligned}$$

To maximize the likelihood, we want to make θ as small as possible subject to the constraint that $x_i \le \theta$ for all i. Thus, the maximum likelihood estimate of θ is the largest observed value,

$$\hat{\theta} = x_{(n)}.$$

c) Find the probability density of the mle, and calculate its mean and variance. Compare the variance, the bias, and the mean squared error to those of the method of moments estimate. The probability density of the largest value drawn from a distribution F_X is given by

$$f_{X_{(n)}}(x_{(n)}) = nF_X(x_{(n)})^{n-1} f_X(x_{(n)}).$$

In this case, the underlying distribution is uniform on $[0, \theta]$, so $f_X(x) = 1/\theta$ and $F_X(x) = x/\theta$, so the density of the largest value (given θ) is

$$f(x_{(n)}|\theta) = \frac{n x_{(n)}^{n-1}}{\theta^n}, \quad 0 \le x_{(n)} \le \theta.$$

Thus,

$$\begin{aligned}
E(\hat{\theta}) &= \int_0^\theta \frac{n x_{(n)}^n}{\theta^n} dx_{(n)} = \frac{n}{n+1}\theta \\
E(\hat{\theta}^2) &= \frac{n x_{(n)}^{n+1}}{\theta^n} dx_{(n)} = \frac{n}{n+2}\theta^2 \\
V(\hat{\theta}) &= \left[\frac{n}{n+2} - \frac{n^2}{(n+1)^2}\right]\theta^2 = \frac{n}{(n+2)(n+1)^2}\theta^2. \\
Bias(\hat{\theta}) &= E(\hat{\theta}) - \theta = -\frac{1}{n+1}\theta \\
MSE(\hat{\theta}) &= V(\hat{\theta}) + Bias(\hat{\theta})^2 = \frac{n}{(n+2)(n+1)^2}\theta^2 + \frac{1}{(n+1)^2}\theta^2 = \frac{2}{(n+2)(n+1)}\theta^2
\end{aligned}$$

The method of moments estimator is unbiased (its expected value is the true value of θ), but the mle is slightly biased downwards here - the true value of θ is almost certainly larger than the largest observed value. Thus, in this case the method of moments estimator seems to perform better. This fails to take into account the uncertainty in the estimator though. The variance of the method of moments estimator is $\theta^2/3n$, and as the estimator is unbiased this is also the mean squared error of this estimator. The mean squared error of the method of moments estimator is the same as that of the mle for $n = 1, 2$, but for larger samples the mse of the mle is smaller, with the difference becoming more dramatic as n increases.

d) Find a modification of the mle that renders it unbiased. In this case, the most obvious modification is a simple multiplicative adjustment: multiplying by $(n+1)/n$.

$$E\left(\frac{n+1}{n}\hat{\theta}\right) = \frac{n+1}{n}E(\hat{\theta}) = \frac{n+1}{n}\frac{n}{n+1}\theta = \theta,$$

so this new estimate is unbiased. Even though the problem doesn't ask for it, let's check the variance and MSE. The variance of the new estimator is

$$V\left(\frac{n+1}{n}\hat{\theta}\right) = \frac{1}{n(n+2)}\theta^2,$$

and as this estimator is unbiased, this is also the MSE. This estimator has a smaller MSE than the mle (and the method of moments estimator) for $n > 1$.

8.57 Let $X_1, \ldots, X_n$ be an i.i.d. sample from a distribution with the density function

$$f(x|\theta) = \frac{\theta}{(1+x)^{\theta+1}}, \quad 0 < \theta < \infty \quad \text{and} \quad 0 \leq x < \infty.$$

Find a sufficient statistic for θ.

$$\begin{aligned} L(\theta) &= \prod_{i=1}^{n} \frac{\theta}{(1+x_i)^{\theta+1}} \\ l(\theta) &= \sum_{i=1}^{n} [\log(\theta) - (\theta+1)\log(1+x_i)] \\ &= n\log(\theta) - (\theta+1)\sum_{i=1}^{n} \log(1+x_i). \end{aligned}$$

Thus, $\sum \log(1+x_i)$ (or, equivalently, $\prod(1+x_i)$) is a sufficient statistic for θ.

8.58 Show that $\prod_{i=1}^{n} X_i$ and $\sum_{i=1}^{n} X_i$ are sufficient statistics for the gamma distribution.

$$\begin{aligned} L(\lambda, \alpha) &= \prod_{i=1}^{n} \frac{\lambda^\alpha}{\Gamma(\alpha)} x_i^{\alpha-1} e^{-\lambda x_i} \\ l(\lambda, \alpha) &= \sum_{i=1}^{n} [\alpha \log(\lambda) - \log(\Gamma(\alpha)) + (\alpha-1)\log(x_i) - \lambda x_i] \\ &= n\alpha\log(\lambda) - n\log(\Gamma(\alpha)) + (\alpha-1)\sum \log(x_i) - \lambda \sum x_i \end{aligned}$$

so by inspection, $\sum \log(x_i)$ and $\sum x_i$ are sufficient for estimating α and λ, so if we know the values of these two functions of the data that's all that we need. Noting that $\sum \log(x_i) = \log(\prod x_i)$, we can state that knowledge of the product is equivalent to knowledge of the sum of the logs, and hence $\prod_{i=1}^{n} x_i$ and $\sum_{i=1}^{n} x_i$ are sufficient statistics for the gamma distribution.