Mean Residual Life Function

By Musie Ghebremichael, Rice University

14th January 2003

Introduction To Survival Concepts

Survival analysis is a loosely defined statistical term that encompasses a variety of statistical techniques for analyzing positive-valued random variables. Typically the value of the random variable is the time to the failure of a physical component(mechanical or electrical) or the time to the death of a bilogical unit (patient,animal, cell, etc.). However, it could be the time to the learning of a skill, or it may not even be a time at all.

Definition 1. *{Survival And Hazard Functions}:Let* $T \geq 0$ *have density* $f(t)$ *and distribution function(d.f)* $F(t)$. *The survival function* $S(t)$ *is*

$$S(t) = 1 - F(t) = P\{T > t\}$$

and the hazard rate or hazard function $h(t)$ *is*

$$h(t) = \frac{f(t)}{S(t)}$$

The hazard rate has the following interpretation

$$h(t)dt = P\{t < T < t + dt\,|T > t\}$$

$$= \quad P\left\{expiring\ in\ (t, t+dt)\left|survived\ past\ t\right.\right\}$$

Integrating $h(t)$,

$$\begin{aligned} \int_0^t h(u)du &= \int_0^t \frac{f(u)}{S(u)}du \\ &= -\ln S(t) \end{aligned}$$

which leads to the important expression

$$S(t) \quad = \quad \exp\left(-\int_0^t h(u)du\right)$$

Let $\{T_n, n \geq 1\}$ *be independently, identically distributed(iid) random variables(representing the lifetimes of individuals) with common distribution* F*. We imagine* F *is unknown and on the basis of a sample* $T_1, T_2, ...T_n$ *we seek to estimate the survival function. The estimator will be the* empirical survival function *defined by*

$$S_n(t) \quad = \quad \frac{1}{n}\sum_{i=1}^{n} I_{\{T_i > t\}} \ \forall\, t \geq 0$$

By the SLLN we get that for each fixed t, $S_n(t) \to S(t)$ a.s. as $n \to \infty$. In fact the convergence is uniform in t, that is

$$\sup_t |S_n(t) - S(t)| \quad \to \quad 0 \; a.s. \;\; as \;\; n \to \infty$$

By the central limit theorem we get that for each fixed t, the empirical process

$$\sqrt{n}\,(S_n(t) - S(t)) \quad \stackrel{d}{\to} \quad \mathbb{N}\,(0, S(t)\,[1 - S(t)])$$

We have the following more stronger result

$$\left\{\sqrt{n}\,(S_n(t) - S(t)) \;\; \forall\, t > 0\right\} \quad \to \quad A\ Guassian\ process$$

Survival Functions And Censoring

What distinguishes survival analysis from other fields of statistiics is censoring. Vaguely speaking, a censored observation contains only partial information about the random variable of interest. We consider three types of censoring. Let $T_1, \ldots\ldots, T_n$ be iid each with d.f. F.

1. *Type I Censoring:* Let t_c be some(preassigned) fixed number which we call the fixed censoring time. Instead of observing $T_1, ..., T_n$(the random variables of interest) we can only observe $Y_1, ..., Y_n$ where

$$Y_i = \begin{cases} T_i & if \quad T_i \leq t_c \\ t_c & if \quad t_c < T_i \end{cases}$$

Notice that the distribution function of Y has positive mass $P\{T > t_c\} > 0$ at $y = t_c$.

2. *Type II Censoring:* Let $r < n$ be fixed, and let $T_{(1)}, ..., T_{(n)}$ be the order statistics of $T_1, ..., T_n$. Observation ceases after the $r - th$ failure so we can observe $T_{(1)}, ..., T_{(n)}$. The full ordered sample is

$$Y_{(i)} = \begin{cases} T_{(i)} & if \quad T_{(i)} \leq T_{(r)} \\ T_{(r)} & if \quad T_{(i)} > T_{(r)} \end{cases}$$

3. *Random Censoring:* Let $C_1, C_2, ...C_n$ be iid each with d.f. G. C_i is the censoring time associated with T_i.

We can only observe $(Y_1, \delta_1), ..., (Y_n, \delta_n)$ where

$$\begin{aligned} Y_i &= \min(T_i, C_i) = T_i \wedge C_i \\ \delta_i &= \begin{cases} 1 & if \quad T_i \le C_i(Uncensored) \\ 0 & if \quad T_i > C_i(Censored) \end{cases} \end{aligned}$$

Notice that $Y_1, ..., Y_n$ are iid with some d.f. H. Also $\delta_i, ..., \delta_n$ contain the censoring information. (In Type I and Type II censoring we also were able to observe which items were censored, but since it was easy to see which ones these were, we did not need to define the $\delta_i' s$ explicitly.

Both Type I and Type II censoring arise in engineering applications. In such situations there is a batch of transistors or tubes; we put them all on test at $t = 0$, and record their times to failure. Some transistors may take a long time to burn out, and we will not want to wait that long to end the experiment. Therefore, we might stop the experiment at a specified time t_c, in which case we have Type I censoring, or we might know beforehand what value of the fixed censoring time is good so we decide to wait until a prespecified fraction $\frac{r}{n}$ of the transistors has burned out, in which case we have Type II censoring. Random censoring arises in medical applications with animal studies or clinical trials. In a clinical trial, patients may enter the study at different times; then each is treated with one of several therapies. We want to observe their lifetimes, but censoring occurs in one of the following forms:

- Lost to follow-up. The patients may decide to move elsewhere; we never see him again.
- Drop out. The therapy may have such bad side effects that it is neccessary to discontinue the treatment. Or the patient may still be in contact (he has not moved), but refuses to continue the treatment.
- Termination of the study.

With random censoring we will assume T_i and C_i are independent. Without this assumption few results are available. It seems justified with random entries to the study and randomly occuring losses to follow-up. However, if the reason for dropping out is related to the course of the therapy, there may well be dependence between T_i and C_i.

The previous types of censoring fall under the heading of *right censoring* : if the random variable of interest is too large, we do not get to observe it completely. There is also *left censoring.* For example, in random left censoring, we can only obsereve $(Y_1, \varepsilon_1), ..., (Y_n, \varepsilon_n)$ where

$$\begin{aligned} Y_i &= \max(T_i, C_i) \\ \varepsilon_i &= I(C_i \leq T_i) \end{aligned}$$

Both right and left censoring are special cases of i*nterval censoring,* in which we may only get to see that the random variable of interest falls in an interval. If T_i is random right censored, we get to observe that T_i falls in the interval $[C_i, \infty)$, and if T_i is random left censored, we get to observe that T_i falls in the interval $(-\infty, C_i]$.There are examples of more general interval censoring.

In contrast to interval censoring there is truncation in which if the random variable of interest falls outside some interval, even its existence is unobserved. For example, suppose we want to get the distribution and expected size of a certain organelle in the cell. Because of limitations on the measuring equipment, if an organelle is below a certain size it can not be detected.

Kaplan-Meir Estimator

Let $X_{1,....,}X_n \overset{iid}{\sim} F$ be the failure time and $C_1, ..., C_n \overset{iid}{\sim} G$ be the censoring times. The data which we observe are are the pairs (t_i, δ_i) $i = 1, ...n$ where $T_i = \min(X_i, C_i)$ and $\delta_i = I(X_i = T_i)$. Suppose $t_1 < t_2 < .... < t_k (k < n)$ are the times at which the actual deaths occur. Let us denote the number of individuals who are at rik just prior to t_i and the number of deaths at t_i by n_i and d_i respectively. The kaplan-Meir estimator(hereafter refered to as K-M) is given by

$$\begin{aligned}
S_{KM}(t) &= \prod_{j:t_j \leq t} \left(1 - \frac{d_j}{n_j}\right) \\
&= \prod_{j=1}^{s} \left(1 - \frac{d_j}{n_j}\right), \;\; for\; x_j \leq s < t_{j+1}
\end{aligned}$$

Properties Of The K-M Estimator: Suppose that the censoring distribution G and the underlying survival function

F do not have jumps in common. Let

$$\begin{aligned} T_F &= \inf\{t : F(t) = 1\} \\ T_G &= \inf\{t : G(t) = 1\} \\ T* &= \min(T_F, T_G) \end{aligned}$$

The K-M estimator has the following propertines:

1.

$$E[S_{KM}(t) \leq S(t)$$

2.

$$\sup_{t \leq T*} |S_{KM}(t) - S(t)| \rightarrow 0 \; as \; n \rightarrow \infty \;\; w.p.1$$

3. Let $T < \infty$ with $S(T) > 0$and suppose that both G and F are continoues distributions. Then

$$\left\{\sqrt{n}\left(S_{KM}(t) - S(t)\right), 0 < t < T\right\}$$

converges weakly to a Guassian process.

The Mean Residual Life Function

The concept of Mean Residual Life (MRL) is based on conditional expectation and has been of much interest in the actuarial sciences, survival studies and reliability theory. A literature survey by Guass and Proschan (1985) traces its histry to the third century A.D. It has been used extensively to model the 'burn-in' problem in reliability theory. Watson and Wells (1961) explore improving MRL of a batch articles through a test run. Defective articles fail early on and are eliminated, leaving the rest of the batch with a longer MRL. The concepts of Increasing Mean Residual Life (IMRL), Decreasing Mean Residual Life (DMRL), New Better than Used (NBU) etc. are used in the social sciences to analyze duration of wars and strikes. In particular, the paper by Bhattachrjee(1982) sets the conditions for a function to be an MRL function. Yang(1978) proposed a nonparametric estimate of MRL function and studies its converegence properties. Hall and Wellner (in an unpublished paper) have reformulated and extended her results.Oakes and Dasu (1990) proposed the Propertional Mean Residual Life (PMRL) model. The PMRL was extended to a regression model with explanatory variables by Maguluri and Zhang (1994). Yuen, Zhu and

Tang (2001) devoloped a goodness of test for the PMRL regression model.

Definition, Properties and Characterizations

We are considering probability models in which a non-negative and continuous random variable T arises. For example, T might reprsent the time to failure of a component, or the repair time of a component, or the time required to perform a service, or the waiting time in a queue. Thus, in typical contexts, the dimension of T is time.

Given that a unit is of age t, the remaining life after time t is random. The expected value of this random residual life is called the Mean Residual Life (MRL) at time t, since the MRL is defined for each time t, we also speak of the MRL function. The expected residual value, or mean residual value of T given that t units of time has elapsed, is denoted by $e_T(t) = e(t)$ and is defined as

$$\begin{aligned} e(t) &= E\left[T - t\,|T > t\right],\ t \geq 0 \\ &= \frac{\int_t^\infty S(u)du}{S(t)} \end{aligned}$$

$$= \frac{\int_t^\infty u f(u) du}{S(t)} - t$$

In terms of the various phenomena which T can represent, $e(t)$ may be called mean residul life, mean residual repair time, and so on. We shall use the designation mean residual life (MRL) in all cases. Specifically, when T represents the lifetime, or time to failure of a component, then $e(0)$ is the expected life of a new component and $e(t)$ is the expected life remaining in a component of age t. Here age t means that the component has survived up to time t and is still operational at time t. When T represents the repair time of a component, then $e(t)$ is the expected time required to complete the repair of a component on which t units of repair time have already been expended.

Several classes of distributions are defined in terms of the MRL:NBUE($e(t) \leq e(0) = \mu \ \ \forall t$),NWUE ($e(t) \geq \mu \ \ \forall t$), IMRL($e \uparrow$), and DMRL ($e \downarrow$). These are larger than the classes IFRA and DFRA respectively($iff \ \ \mu < \infty$).

The probability law of T may be expressed uniquely interms of its density function $f(t)$, its distribution function

$F(t)$, its survival function $S(t)$ and its hazard function $h(t)$. We shall now establish relantioships between $e(t)$ and these functions and show that $e(t)$ is also a unique representation of the probability law of T. The mean value of T is $\mu = e(0)$ and is assumed to be finite.

$$
\begin{aligned}
e(t) &= \frac{\int_t^\infty S(u)du}{S(t)} \\
&= \frac{\int_0^\infty S(u)du - \int_0^t S(u)du}{S(t)} \\
&= \frac{1}{S(t)}\left[\mu - \int_0^t S(u)du\right] \qquad (1)
\end{aligned}
$$

Multiplying (1) by $S(t)$ and differentiating with respect to t yields

$$
e\prime(t)S(t) + e(t)S\prime(t) = -S(t)
$$

and after arranging terms

$$
\frac{1 + e\prime(t)}{e(t)} = h(t) \qquad (2)
$$

where

$$h(t) \quad = \quad \frac{f(t)}{S(t)} = -\frac{S\prime(t)}{S(t)}$$

is the hazard function. Since both $e(t)$ and $h(t)$ must be non-negative for all t it follows from (2) that

$$e\prime(t) \quad \geq \quad -1$$

The survival function is expressed interms of $h(t)$ as

$$S(t) \quad = \quad \exp\left[-\int_0^t h(x)dx\right]$$

and substituting (2) we obtain