Stat 331/Elec 331, Solutions to Homework 2

1a. $P(A) = 1/2, P(B) = 2/3, P(C) = 3/36$ and $P(A \cap B \cap C) = 1/36$.

b. No, since for instance A and C are not independent; $P(A \cap C) = 3/36 \neq P(A)P(C)$ (actually, if C occurs, A must also occur). That $P(A \cap B \cap C) = P(A)P(B)P(C)$ is not enough for independence.

2a. Let A_j be the event that we get the ace of spades in the jth draw. Since we draw with replacement and independently, the event that we get the ace of spades for the first time in the kth draw is the event $A_1^c \cap A_2^c \cap ... \cap A_{k-1}^c \cap A_k$ and by independence the probability of this is $P(A_1^c) \cdot ... \cdot P(A_{k-1}^c)P(A_k) = (51/52)^{k-1}1/52$ (a geometric distribution with $p = 1/52$).

b. The probability that the ace of spades appears in the first trial is 1/52, hence $P(X = 1) = 1/52$. That it appears in the second trial means that we must draw something else in the first trial, which has probability 51/52, and then draw the ace of spades from the remaining 51 cards which has probability 1/51. Hence $P(X = 2) = 51/52 \cdot 1/51 = 1/52$ (formally we have used the formula $P(C \cap D) = P(D|C)P(C)$ here). Similarly we will have $P(X = k) = 1/52$ for $k = 1, 2, ..., 52$.

Don't confuse conditional and unconditional probabilities here: the probability that the ace of spades appears in the 51st trial is 1/52 but the conditional probability that it appears in the 51st trial *given that it has not appeared before* is 1/2.

3. For each function f, check that $f \geq 0$ and $\int_{-\infty}^{\infty} f(x)dx = 1$.

a. This is a pdf. To compute the distribution function, first take $-1 \leq x \leq 0$. Then $\int_{-1}^{x} f(t)dt = \int_{-1}^{x} -tdt = -[t^2/2]_{-1}^{x} = 1/2(1 - x^2)$. For $0 \leq x \leq 1$ we have and $\int_{-1}^{x} f(t)dt = 1/2 + \int_0^x tdt = 1/2(1 + x^2)$. Hence

$$F(x) = \begin{cases} 1/2(1 - x^2) & \text{if } -1 \leq x \leq 0 \\ 1/2(1 + x^2) & \text{if } 0 \leq x \leq 1 \end{cases}$$

b. This is not a pdf since it is negative between 0 and 1 (that it has integral equal to 1 does not help in this case).

c. This is a pdf with $F(x) = x + 1, -1 \leq x \leq 0$ (the uniform distribution on $[-1, 0]$).

d. This is a pdf with $F(x) = \int_1^x (1/t^2)dt = [1/t]_1^x = 1 - 1/x, x \geq 1$.

4. For Y, first note that the range is $(0, 1)$. Take an x in $(0, 1)$ and start by finding the cdf:

$$F_Y(x) = P(Y \leq x) = P(\sqrt{X} \leq x) = P(X \leq x^2) = F_X(x^2) = x^2$$

since $X \sim$unif$(0, 1)$. Differentiation gives $f_Y(x) = 2x, 0 < x < 1$..

For Z, note that the range is $(1, \infty)$. Take an $x > 1$. Then

$$F_Z(x) = P(Z \leq x) = P(\frac{1}{X} \leq x) = P(X \geq \frac{1}{x}) = 1 - \frac{1}{x}$$

since X is uniform on $(0, 1)$ (note that $x > 1$ and therefore $1/x$ is between 0 and 1). Hence Z has pdf

$$f_Z(x) = \frac{1}{x^2}, \quad x > 1.$$

5. Let F be the cdf of X. First note that $F(0) = P(X \leq 0) = P(X = 0) = 0.2$. For $x > 0$, let $A = \{X = 0\}$ and apply the Law of Total Probability to get

$$F(x) = P(X \leq x) = P(X \leq x|A)P(A) + P(X \leq x|A^c)P(A^c).$$

Now, $P(A) = 0.2, P(A^c) = 0.8, P(X \leq x|A) = 1$ and $P(X \leq x|A^c) = x/30$ and hence

$$F(x) = \begin{cases} 0.2 + 0.8x/30 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Since the range of X is uncountable, X is not discrete and since the cdf of X is not continuous at 0, X is not continuous. It is a mixture of a discrete part at 0 and a continuous part on (0,30).