Stat 331/Elec 331, Solutions to Homework 3

1. Since $V = 4\pi R^3/3$, we get

$$E[V] = \frac{4\pi}{3}\int_0^1 r^3 f_R(r)dr = \frac{4\pi}{3}\int_0^1 r^3 dr = \frac{4\pi}{3}\frac{1}{4} = \frac{\pi}{3}$$

and since $V^2 = (4\pi/3)^2 R^6$ we get

$$E[V^2] = (\frac{4\pi}{3})^2 \int_0^1 r^6 dr = \frac{16\pi^2}{63}$$

which gives

$$Var[V] = E[V^2] - (E[V])^2 = \frac{\pi^2}{7}.$$

2a. $P(X \leq m) = F_X(m) = \frac{m-a}{b-a} = 1/2$ gives $m = \frac{a+b}{2} = \mu$.

2b. $P(X \leq m) = F_X(m) = 1 - e^{-\lambda m} = 1/2$ gives $m = \frac{\log 2}{\lambda} = \mu \log 2$.

2c. $P(X \leq m) = F_X(m) = \int_1^m \frac{1}{x^2}dx = 1 - \frac{1}{m} = 1/2$ gives $m = 2$. Here $\mu = \infty$.

3.

$$F(x) = 1 - e^{(-\int_0^x r(t)dt)} = 1 - e^{(-\log(1+x))} = 1 - \frac{1}{1+x}$$

and the pdf is $f(x) = F'(x) = 1/(1+x)^2, x \geq 0$.

4.

$$r(t) = \frac{f(t)}{1-F(t)} = \frac{1}{1-t},\ 0 < t < 1.$$

As t approaches 1, $r(t)$ goes to infinity. If we think of X as a lifelength, we know that it cannot be more than 1, so the closer its age comes to 1, the more likely is it that it will die soon, that is, the risk of death increases indefinitely.

5a. Since

$$\{X = k\} = \{X = k, X \geq k\},$$

we get

$$P(X = k) = P(X = k, X \geq k)$$

and hence

$$r(k) = \frac{P(X = k)}{P(X \geq k)} = \frac{P(X = k, X \geq k)}{P(X \geq k)} = P(X = k | X \geq k).$$

b. Here

$$P(X \geq k) = \sum_{j=k}^{6} P(X = j) = \sum_{j=k}^{6} \frac{1}{6} = \frac{7-k}{6}$$

and we get failure rate function

$$h(k) = \frac{P(X = k)}{P(X \geq k)} = \frac{1}{7-k}, \quad k = 1, ..., 6.$$

The failure rate function thus increases from 1/6 to 1 whereas the pmf is constant equal to 1/6 on the range 1,2,...,6. The difference is that the failure rate function gives *conditional probabilities* but the pmf gives unconditional probabilities. For example, $P_X(5)$ is the probability that X equals 5, which is 1/6, but $h(5)$ is the probability that X equals 5 *given that it is at least 5* which is 1/2 since we then know that we have either 5 or 6.

In a lifelength interpretation, consider the following macabre example: you are about to be executed a particular week. The executioner rolls a die on Sunday to determine which of the days Monday through Saturday you will be executed (1=Monday, 2=Tuesday etc) but does not tell you. Each day is thus equally probable, reflected in the pmf which is 1/6 for each value 1,2,...,6. The failure rate function on the other hand, is the conditional probability that you will be executed on a particular day, given that you have survived so far. For example, when you wake up on Monday morning, there is a 1/6 probability that you will be executed. If you survive Monday, when you wake up on Tuesday morning, the failure rate function has gone up to 1/5 which is the probability that you will die that day. The failure rate function keeps going up and its value on Saturday is 1, the probability that you will die Saturday if you have survived Monday-Friday.