Stat 331/Elec 331, Solutions to Midterm Exam

1a. $1 = \int_0^1 f(x)dx = \int_0^1 cx^2 dx = c/3 \Rightarrow c = 3.$

b. $E[X] = \int_0^1 xf(x)dx = 3\int_0^1 x^3 dx = 3/4.$

$E[X^2] = \int_0^1 x^2 f(x)dx = 3/5$ which gives $Var[X] = 3/5 - (3/4)^2 = 1/30.$

c. Start with the cdf of $Y, F_Y : F_Y(x) = P(Y \leq x) = P(\sqrt{X} \leq x) = P(X \leq x^2) = \int_0^{x^2} f(t)dt = x^6$. Hence

$$f_Y(x) = F_Y'(x) = 6x^5 \quad 0 < x < 1.$$

d. $E[Y] = \int_0^1 xf_Y(x)dx = 6/7$. Further, $E[Y^2] = \int_0^1 x_Y^2(x)dx = 3/4$ so $Var[Y] = 3/4 - (6/7)^2 = 3/196.$

2a. False. Let X be any continuous random variable. Then $P(X = x) = 0$ for all x.

b. True. $f(x) = P(X = x) \leq 1$ by the probability axioms.

c. False. If $X \sim \text{unif}(0, 1/2)$ then $f(x) \equiv 2$.

d. True. $P(2X \leq x) = P(X \leq x/2) = 1 - e^{-\lambda x/2}$, the cdf of $\exp(\lambda/2)$.

e. True. $P(|X| \leq x) = P(-x \leq X \leq x) = \int_{-x}^{x} \frac{1}{2}dt = x.$

f. False. $P(2X = 1) = P(X = 1/2) = 0$ but in a $\text{bin}(2n, p)$ distribution this probability is $2np(1-p)^{2n-1}$.

3a. Let $S = \{$ 1 was sent $\}$ and $R = \{$ 1 was received $\}$. Then by Bayes' formula

$$P(S|R) = \frac{P(R|S)P(S)}{P(R|S)P(S) + P(R|S^c)P(S^c)}$$

where $P(R|S) = 0.90, P(R|S^c) = 0.10, P(S) = 1/3$ and $P(S^c) = 2/3$ so $P(S|R) = 0.82.$

b. $P(\text{received incorrectly}) = 1 - P(\text{received correctly}) = 1 - 0.90 \cdot 0.90$ (by independence) $= 0.19$.

c. From **a** we get that $P(R|S) = 0.82$ for a "1" and in the same way the corresponding probability for "0" is 0.95. By independence the probability asked for is $0.82 \cdot 0.95 = 0.78$.

4. Let the arrival times be X and Y. The probability we are looking for is $P(|X - Y| > 10)$ where X and Y are independent and uniform on $[0, 30]$ and $[0, 45]$ respectively. By independence, the joint density is $f(x, y) = 1/30 \cdot 1/45 = 1/1350, 0 \le x \le 30, 0 \le y \le 45$. Let

$$A = \{(x, y) : 0 \le x \le 30, 0 \le y \le 45, |x - y| > 10\}$$

(draw a picture). Then $P(|X - Y| > 10) = \int_A f(x, y)dxdy = 1/1350 \times$the area of $A = 1/1350 \cdot 800/1350 = 0.59$.

5a. $f(x, y) = f_Y(y|x)f_X(x) = x, \quad 0 < x < 1, 0, y, 1/x$.

b. $P(X > Y) = \int_0^1 \int_0^x xdydx = 1/3$.

c. The range of Y is $(0, \infty)$. For $0 < y \le 1$, we get

$$f_Y(y) = \int_0^1 f(x, y)dx = \int_0^1 xdx = 1/2$$

and for $y > 1$,

$$f_Y(y) = \int_0^{1/y} f(x, y)dx = \int_0^{1/y} xdx = 1/(2y^2)$$

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