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THE $x \log x$ CONDITION FOR GENERAL BRANCHING PROCESSES

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Abstract

The $x \log x$ condition is a fundamental criterion for the rate of growth of a general branching process, being equivalent to non-degeneracy of the limiting random variable. In this paper we adopt the ideas from Lyons, Pemantle and Peres (1995) to present a new proof of this well-known theorem. The idea is to compare the ordinary branching measure on the space of population trees with another measure, the *size-biased* measure.

Keywords: General branching process; $x \log x$ condition; immigration

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1. Introduction

The $x \log x$ condition is a fundamental concept in the theory of branching processes, being the necessary and sufficient condition for a supercritical branching process to grow as its mean. In a Galton–Watson process with offspring mean $m = E[X] > 1$, let Z_n be the number of individuals in the nth generation and let $W_n = Z_n/m^n$. Then W_n is a non-negative martingale and hence $W_n \to W$ for some random variable W. The *Kesten–Stigum theorem* is as follows.

Theorem 1.1. *If* $E[X \log X] < \infty$ *then* $E[W] = 1$*; if* $E[X \log X] = \infty$ *then* $W = 0$ *a.s.*

It can further be shown that $P(W = 0)$ must either be 0 or equal the extinction probability and hence $E[X \log X] < \infty$ implies that $W > 0$ exactly on the set of non-extinction (see for example Athreya and Ney (1972)).

The analogue for general single-type branching processes appears in Jagers and Nerman (1984) and a partial result for general multitype branching processes in Jagers (1989). Lyons *et al.* (1995) give a new, very elegant proof of the Kesten–Stigum theorem based on comparisons between the Galton–Watson measure and another measure, the *size-biased* Galton–Watson measure, on the space of progeny trees. Our aim is to further develop these ideas to general branching processes. To make this paper self-contained, we give a short review of general branching processes in the next section. As in the Galton–Watson case, it turns out that a certain branching process with *immigration* is crucial in the proof; for that purpose we briefly discuss processes with immigration in Section 3, following Olofsson (1996).

So called size-biased processes are described in Section 4, and in Section 5 the size-biased measure on the space of population trees is constructed. Not much effort is then needed to conclude the proof in Section 6.

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2. The $x \log x$ condition for general branching processes

We start by giving a quick description of general branching processes, following Jagers and Nerman (1984). Individuals are identified by descent; the ancestor is denoted by 0, the individual $x = (x_1, \ldots, x_n)$ belongs to the nth generation and is the x_nth child of the x_{n-1}th child of ... of the x_1th child of the ancestor. This gives the *population space*

$$I = \bigcup_{n=0}^{\infty} N^n,$$

the set of all individuals.

With each individual x, we associate its *birth-time* τ_x and *reproduction process* ξ_x. The ξ_x are i.i.d. copies of ξ, where $\xi(a)$ is the number of children an individual begets before age a. We assume that $\xi(0) \equiv 0$ and $E[\xi(a)] < \infty$ for all $a \geq 0$. The ancestor is assumed to be born at time $\tau_0 \equiv 0$.

A *general branching process* (or population) is a probability space $(\Omega, \mathcal{A}, P)$, where $\omega \in \Omega$ is a tree describing family relations between individuals and their real time evolution, i.e. reproduction processes. Hence an element ω is of the form $(\xi_x)_{x \in I}$. If Q is a probability measure on the set of realizations of point processes with a finite number of points, then $P = Q^I$, since individual reproductions are i.i.d.

Usually the construction is made more general, taking into account not only the reproduction of an individual, but rather her entire *life*. The reproduction process is then considered as one of many possible random objects on the *life space*. The simpler description given here is quite enough for our purposes, though.

To count, or measure, the population, *random characteristics* are used. A random characteristic is a real-valued process χ, where $\chi(a)$ gives the contribution of an individual of age a. We assume that χ is non-negative and vanishing for negative a (no individual contributes before her birth). The χ-value pertaining to the individual x is denoted by χ_x; the χ_x, $x \in I$, are then i.i.d. random objects. The *χ-counted population*, Z_t^χ is defined as

$$Z_t^\chi = \sum_{x \in I} \chi_x(t - \tau_x),$$

the sum of the contributions of all individuals (at time t the individual x is of age $t - \tau_x$). The simplest example of a random characteristic is $\chi(t) = I_{R_+}(t)$, which is 0 before you are born and 1 afterwards. Then Z_t^χ is the number of individuals born up to time t.

The growth of the population is determined by the *Malthusian parameter*, α. Denote the Laplace transform of ξ by ξ^λ, i.e.

$$\xi^\lambda = \int_0^\infty \mathrm{e}^{-\lambda t} \xi(\mathrm{d}t),$$

and define the Malthusian parameter through

$$E[\xi^\alpha] = 1,$$

where we assume that $0 < \alpha < \infty$, the *supercritical case*. Also assume that the *stable age of child-bearing*, β, defined through

$$\beta = \int_0^\infty t \mathrm{e}^{-\alpha t} E[\xi(\mathrm{d}t)], \tag{2.1}$$

is finite. The fundamental convergence result is

$$e^{-\alpha t} Z_t^{\chi} \to \frac{E[\chi^{\alpha}]}{\alpha\beta} W, \tag{2.2}$$

as $t \to \infty$. Under different sets of conditions different modes of convergence may be obtained. We will not mention this further, the emphasis being on the properties of the limiting random variable W. This variable is the limit of the *intrinsic martingale*, W_t, introduced in Nerman (1981). Denote x's mother by mx and let

$$\mathcal{I}_t = \{x : \tau_{mx} \le t < \tau_x\},$$

the set of individuals whose mothers are born before time t, but who themselves are born after t. Now let

$$W_t = \sum_{x \in \mathcal{I}_t} e^{-\alpha \tau_x},$$

the individuals in $\mathcal{I}_t$ summed with time dependent weights. If $\mathcal{F}_t$ is the σ-algebra generated by the reproductions of the individuals not stemming from $\mathcal{I}_t$ (the *pre-*$\mathcal{I}_t$*-*σ fields, see Jagers (1989) for details) then $\{W_t, \mathcal{F}_t\}$ is a real time non-negative martingale with $E[W_t] = 1$.

By (2.2), χ enters asymptotically only through a constant. All the randomness is captured in W and the $x \log x$ theorem now takes the following form.

Theorem 2.1. *If* $E[\xi^{\alpha} \log^{+} \xi^{\alpha}] < \infty$ *then* $E[W] = 1$*; if* $E[\xi^{\alpha} \log^{+} \xi^{\alpha}] = \infty$ *then* $W = 0$ *a.s.*

As in the Galton–Watson case $E[W] = 1$ implies that $W > 0$ on the set of non-extinction.

Also note that in the Galton–Watson case, $\xi^{\alpha} = Xe^{-\alpha}$ which gives $\alpha = \log m$ and we recognize the Kesten–Stigum theorem.

3. Processes with immigration

Now consider a general branching process where new individuals $v_1, v_2, \ldots$ immigrate according to some point process η with points $(\tau_{v_1}, \tau_{v_2}, \ldots)$. Immigrating individuals initiate independent branching processes according to the population law P. If we start with no individuals at time 0 and let $Z_t^{\chi}(k)$ denote the process started by v_k at time τ_{v_k}, we can describe the resulting process, denoted by $\widetilde{Z}_t^{\chi}$, as the superposition

$$\widetilde{Z}_t^{\chi} = \sum_{k=1}^{\eta(t)} Z_{t-\tau_{v_k}}(k), \tag{3.1}$$

a *general branching process with immigration*. Considering τ_{v_k} as v_k's birth time, the definitions of $\mathcal{I}_t$ and W_t remain the same when there is immigration but there is one fundamental difference. In a process without immigration the sets $\mathcal{I}_t$, $t \ge 0$, are *covering* in the sense that, with $t_1 \le t_2$, $\mathcal{I}_{t_2}$ *stems from* $\mathcal{I}_{t_1}$, i.e. any individual in $\mathcal{I}_{t_2}$ either itself belongs to $\mathcal{I}_{t_1}$ or has an ancestor in it. When there is immigration this is no longer true since an individual in $\mathcal{I}_{t_2}$ can also stem from an individual who immigrated between t_1 and t_2 and the martingale property no longer holds. A useful fact is that with the special characteristic

$$\chi^{*}(t) = e^{\alpha t} \int_{(t,\infty)} e^{-\alpha u} \xi(\mathrm{d}u), \qquad t \ge 0$$

($\chi^*(t) = 0$ for $t < 0$) we have $W_t = \mathrm{e}^{-\alpha t} Z_t^{\chi^*}$. Agree to use the notation $\widetilde{W}_t$ when there is immigration and W_t when there is not and apply (3.1) to obtain

$$\widetilde{W}_t = \sum_{k=1}^{\eta(t)} \mathrm{e}^{-\alpha \tau_{v_k}} W_{t-\tau_{v_k}}.$$

As mentioned above, this is no longer a martingale. The asymptotics of $\widetilde{W}_t$ in particular and of $\widetilde{Z}_t^{\chi}$ in general is treated (in the more general context of a multitype process with arbitrary type space) in Olofsson (1996). From that reference we need only the following lemma.

Lemma 3.1. *On* $\{\eta^\alpha < \infty\}$ *there exists an a.s. finite random variable* $\widetilde{W}$ *such that* $\widetilde{W}_t \to \widetilde{W}$ *a.s. as* $t \to \infty$.

Note that, since η has points $(\tau_{v_1}, \tau_{v_2}, \dots)$, its Laplace transform becomes

$$\eta^\alpha = \int_0^\infty \mathrm{e}^{-\alpha t} \eta(\mathrm{d}t) = \sum_{k=1}^{\infty} \mathrm{e}^{-\alpha \tau_{v_k}},$$

a representation that will be used in what follows.

4. Size-biased processes

A crucial concept for the Lyons–Pemantle–Peres proof is that of *size-biased trees*. In the Galton–Watson case these are constructed with the aid of so-called *size-biased random variables* which are defined as follows. Let X be a non-negative, integer-valued random variable with $P(X = k) = p_k$ and $E[X] = m$. A random variable $\widetilde{X}$ is said to have the *size-biased distribution of* X if

$$P(\widetilde{X} = k) = \frac{kp_k}{m}.$$

A size-biased Galton–Watson tree is constructed in the following way. Let X be the number of children and let $\widetilde{X}$ have the size-biased distribution of X. Start with a number $\widetilde{X}_0$ of individuals. Pick one of these at random, call her v_1 and give her a size-biased number $\widetilde{X}_1$ of children. Give the *other* individuals ordinary Galton–Watson descendant trees. Pick one of v_1's children at random, give her a size-biased number $\widetilde{X}_2$ of children, give her sisters ordinary Galton–Watson descendant trees and so on. The resulting tree is called a size-biased Galton–Watson tree. Indeed, with $W_n = Z_n/m^n$, the number of individuals in the nth generation normed by its expectation, GW_n denoting the ordinary Galton–Watson measure restricted to the n first generations and $\widetilde{GW}_n$ denoting the size-biased measure that arises from the above construction, the relation

$$\widetilde{GW}_n = W_n GW_n, \tag{4.1}$$

holds. Thus it is the martingale W_n that size-biases the tree and we note that the size-biased measure $\widetilde{GW}$ on the set of trees gives mass zero to the set of finite trees, i.e. extinct processes.

General branching processes require a more general concept. For that purpose, note that $\widetilde{X}$ has the size-biased distribution of X if and only if

$$P(\widetilde{X} = k) = E\left[\frac{X}{m}; X = k\right].$$

In a general process X is replaced by the reproduction process ξ, the size of which is properly measured by its Laplace transform, ξ^α. We denote by Γ the set of realizations of point processes on R_+ with a finite number of points, equip it with some appropriate σ-algebra $\mathcal{G}$ and make the following definition.

Definition 4.1. The point process $\tilde{\xi}$ is said to have the size-biased distribution of ξ if

$$P(\tilde{\xi} \in A) = E[\xi^\alpha; \xi \in A],$$

for every $A \in \mathcal{G}$.

An immediate consequence of this definition is the following.

Lemma 4.1. *If $\tilde{\xi}$ has the size-biased distribution of ξ then*

$$E[g(\tilde{\xi})] = E[\xi^\alpha g(\xi)],$$

for every measurable function $g : \Gamma \to \mathbb{R}$.

Proof.

$$\begin{aligned} E[g(\tilde{\xi})] &= \int_\Gamma g(\gamma) E[\xi^\alpha; \xi \in \mathrm{d}\gamma] \\ &= \int_\Gamma \gamma^\alpha g(\gamma) P\xi^{-1}(\mathrm{d}\gamma) = E[\xi^\alpha g(\xi)]. \end{aligned}$$

Note that a size-biased point process always contains points since

$$P(\tilde{\xi}(\infty) = 0) = E[\xi^\alpha; \xi(\infty) = 0] = 0.$$

Also note that in the Galton–Watson process, $\xi(\mathrm{d}t) = \delta_1(\mathrm{d}t)X$, and hence $\xi^\alpha = \mathrm{e}^{-\alpha} X = X/m$ (recall the definition of α which here reduces to $m\mathrm{e}^{-\alpha} = 1$).

Size-biased processes are not new in the theory of general branching processes. In fact they appear automatically in the *stable population*, described in Jagers and Nerman (1984). We will return to this fact in the next section.

5. The size-biased measure

Now consider a general branching process without immigration. For a fixed $\omega \in \Omega$, let $[\omega]_t$ denote the set of trees that coincide with ω up to time t. If x is an individual in $\mathcal{I}_t$ then let $[\omega; x]_t$ denote the set of trees with distinguishable paths, such that the tree is in $[\omega]_t$, the path starts from the root, does not backtrack and goes through x. Assume that the ancestor reproduces according to $\gamma \in \Gamma$ and denote the descendant trees of her children $\omega^{(1)}, \omega^{(2)}, \ldots, \omega^{(k)}$, where $k = \gamma(\infty)$. Suppose that x belongs to $\omega^{(i)}$, i.e. $x = iy$ for some y. Then clearly the population law P satisfies the recursion

$$\begin{aligned} \mathrm{d}P[\omega]_t &= P\xi^{-1}(\mathrm{d}\gamma) \prod_{j=1}^{\gamma(t)} \mathrm{d}P[\omega^{(j)}]_{t-\tau_j(\gamma)} \\ &= \gamma^\alpha P\xi^{-1}(\mathrm{d}\gamma) \frac{1}{\gamma^\alpha} \mathrm{d}P[\omega^{(i)}]_{t-\tau_i(\gamma)} \prod_{j \neq i} \mathrm{d}P[\omega^{(j)}]_{t-\tau_j(\gamma)} \end{aligned}$$

where the '$\mathrm{d}\gamma$' is understood but suppressed on the left hand side.

We shall in a moment construct a measure $\widetilde{P}_*$ on the set of infinite trees with infinite distinguishable paths, such that

$$\mathrm{d}\widetilde{P}_*[\omega; x]_t = \gamma^\alpha P\xi^{-1}(\mathrm{d}\gamma)\frac{\mathrm{e}^{-\alpha\tau_i(\gamma)}}{\gamma^\alpha}\mathrm{d}\widetilde{P}_*[\omega^{(i)}; y]_{t-\tau_i(\gamma)}\prod_{j\neq i}\mathrm{d}P[\omega^{(j)}]_{t-\tau_j(\gamma)}. \tag{5.1}$$

Since $\tau_y(\omega^{(i)}) = \tau_x(\omega) - \tau_i(\gamma)$ it is easily obtained that

$$\mathrm{d}\widetilde{P}_*[\omega; x]_t = \mathrm{e}^{-\alpha\tau_x(\omega)}\mathrm{d}P[\omega]_t.$$

The projection $\widetilde{P}$ of $\widetilde{P}_*$ onto the set of trees then satisfies

$$\mathrm{d}\widetilde{P}[\omega]_t = \sum_{x\in\mathcal{I}_t}\mathrm{d}\widetilde{P}_*[\omega; x]_t = W_t(\omega)\mathrm{d}P[\omega]_t, \tag{5.2}$$

which is the analogue of (4.1). The expression for $\mathrm{d}\widetilde{P}_*[\omega; x]_t$ suggests the following construction.

Let $\tilde{\xi}$ be a point process which has the size-biased distribution of ξ, i.e.

$$P(\tilde{\xi} \in A) = E[\xi^\alpha; \xi \in A] = \int_A \gamma^\alpha P\xi^{-1}(\mathrm{d}\gamma).$$

Start with the ancestor, now called v_0, give her a reproduction process $\gamma = \tilde{\xi}_0$ where $\tilde{\xi}_0$ has the distribution of $\tilde{\xi}$. Pick one of the children so that the kth child is chosen with probability $\mathrm{e}^{-\alpha\tau_k(\gamma)}/\gamma^\alpha$. Call this child v_1, give her a size-biased reproduction $\tilde{\xi}_1$ and give her sisters independent descendant trees, each following the law P. Continue in this way and define the measure $\widetilde{P}_*$ to be the joint distribution of the random tree and the random path $(v_0, v_1, \dots)$; then $\widetilde{P}_*$ satisfies (5.1).

It should here be noted that the size-biased measure $\widetilde{P}$ arises by going backwards in the *stable population* described in Jagers and Nerman (1984). Indeed, if we consider *Ego* as the ancestor, the process in which her *mother* was born has the size-biased distribution of the reproduction process. Continuing backwards in this way, regarding consecutive mothers as the individuals selected to form the path and aunts as the remaining children, the resulting measure is $\widetilde{P}$.

The individuals off the path $(v_0, v_1, \dots)$ constitute a general branching process with immigration (the immigrants being the *children* of $v_0, v_1, \dots$, *not* $v_0, v_1, \dots$ themselves). To describe the immigration process, let $I_{j,k}$ be the indicator of the event that v_{j-1}'s kth child is *not* chosen to be v_j and denote the kth point in $\tilde{\xi}$ by $\tau_k(\tilde{\xi})$. The immigration process η is

$$\eta(\mathrm{d}t) = \sum_{j,k}\delta_{\tau_k(\tilde{\xi}_{v_j})}(\mathrm{d}t - \tau_{v_j})I_{j,k},$$

which has Laplace transform

$$\eta^\alpha = \sum_{j,k}\mathrm{e}^{-\alpha\tau_{v_j}}\mathrm{e}^{-\alpha\tau_k(\tilde{\xi}_{v_j})}I_{j,k}.$$

As in Lyons *et al.* (1995) the following simple lemma plays a key role.

Lemma 5.1. *Let $X_1, X_2, \ldots$ be non-negative i.i.d. random variables with expectation μ. Then*

$$\limsup_{n\to\infty} \frac{X_n}{n} = \begin{cases} 0 & \text{if } \mu < \infty \\ \infty & \text{if } \mu = \infty. \end{cases}$$

We can now state the immigration theorem.

Theorem 5.1. *Consider a general branching process with immigration process η as above. If $E[\log^+ \tilde{\xi}^\alpha] < \infty$, then $\lim_{t\to\infty} \widetilde{W}_t$ exists and is finite a.s. while if $E[\log^+ \tilde{\xi}^\alpha] = \infty$, then $\limsup_{t\to\infty} \widetilde{W}_t = \infty$ a.s.*

Proof. First assume that $E[\log^+ \tilde{\xi}^\alpha] < \infty$. From the construction of the immigration process we see that τ_{v_j} is a sum of j i.i.d. random variables $T_1, T_2, \ldots, T_j$ say, with expectation

$$\begin{aligned} E[T_1] &= \int_\Gamma \sum_{m=1}^{\infty} \tau_m(\gamma) \frac{\mathrm{e}^{-\alpha\tau_m(\gamma)}}{\gamma^\alpha} \gamma^\alpha P\xi^{-1}(\mathrm{d}\gamma) \\ &= \int_\Gamma \sum_{m=1}^{\infty} \tau_m(\gamma) \mathrm{e}^{-\alpha\tau_m(\gamma)} P\xi^{-1}(\mathrm{d}\gamma) = \int_0^\infty t\,\mathrm{e}^{-\alpha t} E[\xi(\mathrm{d}t)] = \beta < \infty, \end{aligned}$$

by (2.1). Hence we have that $\tau_{v_j} \sim \beta j$ as $j \to \infty$ so that, for any $\epsilon > 0$, $\mathrm{e}^{-\alpha\tau_{v_j}} \le (\mathrm{e}^{-\alpha(\beta-\epsilon)})^j$ a.s. for j large enough. Now note that

$$\begin{aligned} \eta^\alpha &\le \sum_{j,k} \mathrm{e}^{-\alpha\tau_{v_j}} \mathrm{e}^{-\alpha\tau_k(\tilde{\xi}_{v_j})} \\ &= \sum_{j=1}^{\infty} \mathrm{e}^{-\alpha\tau_{v_j}} \tilde{\xi}^\alpha_{v_j} < \infty \text{ a.s.}, \end{aligned}$$

since, by Lemma 5.1, the $\tilde{\xi}^\alpha_{v_j}$ grow subexponentially. We can now apply Lemma 3.1 to conclude that $\lim_{t\to\infty} \widetilde{W}_t$ exists and is finite a.s.

Now assume that $E[\log^+ \tilde{\xi}^\alpha] = \infty$. Recall the set $\mathcal{I}_t$ and note that if an individual is born at time t, then all her children belong to $\mathcal{I}_t$. Since v_n's kth child is born at time $\tau_{v_n} + \tau_k(\tilde{\xi}_{v_n})$ we obtain

$$\begin{aligned} \widetilde{W}_{\tau_{v_n}} &= \sum_{x\in\mathcal{I}_{\tau_{v_n}}} \mathrm{e}^{-\alpha\tau_x} \\ &\ge \mathrm{e}^{-\alpha\tau_{v_n}} \sum_{k=1}^{\infty} \mathrm{e}^{-\alpha\tau_k(\tilde{\xi}_{v_n})} I_{n,k}. \end{aligned}$$

Since, for each n, $I_{n,k}$ is zero for only one k, we further obtain that

$$\widetilde{W}_{\tau_{v_n}} \ge \mathrm{e}^{-\alpha\tau_{v_n}} (\tilde{\xi}^\alpha_{v_n} - 1)$$

and by Lemma 5.1, $\limsup_{t\to\infty} \widetilde{W}_t = \infty$.

6. Proof of the $x \log x$ theorem

Proof of Theorem 2.1. Let P_t and $\widetilde{P}_t$ be the restrictions of P and $\widetilde{P}$ to $\mathcal{F}_t$. Then, by (5.2),

$$\frac{\mathrm{d}\widetilde{P}_t}{\mathrm{d}P_t} = W_t.$$

Now let $W = \limsup_{t\to\infty} W_t$ (so that it is defined everywhere; clearly $W = \lim_{t\to\infty} W_t$ P-a.s.) and let E denote expectation with respect to P. The following holds (see Durrett (1991), p. 210):

$$\widetilde{P} \ll P \iff W < \infty \ \widetilde{P}\text{-a.s.} \iff E[W] = 1,$$

and

$$\widetilde{P} \perp P \iff W = \infty \ \widetilde{P}\text{-a.s.} \iff E[W] = 0.$$

If $E[\xi^\alpha \log^+ \xi^\alpha] < \infty$ then, by Lemma 4.1, $E[\log^+ \widetilde{\xi}^\alpha] < \infty$. Therefore $W < \infty$ $\widetilde{P}$-a.s. by Theorem 5.1 and hence $E[W] = 1$.

Conversely, if $E[\xi^\alpha \log^+ \xi^\alpha] = \infty$ then $W = \infty$ $\widetilde{P}$-a.s. so that $E[W] = 0$ and hence $W = 0$ P-a.s.

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