STAT 552 Homework 3

Due date: In class on Thursday, October 16, 2003

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8. A Markov chain has state space $S = \{1, 2, \ldots, 8\}$. Starting from $X_0 = 1$, the chain moves in each step from its current state j to any of the larger states $\{k : k > j\}$ with equal probability. State 8 is absorbing.
   (a) Compute the transition matrix.
   (b) Decompose the state space into recurrent and transient classes.
   (c) Find the expected number of steps to reach state 8.

9. Let N_j denote the number of visits to the state j after time 0, i.e.,
$$N_j := \sum_{n=1}^{\infty} 1_{[X_n=j]} \tag{1}$$
   (a) Show that
$$P_i[N_j \geq k] := P[N_j \geq k | X_0 = j] = f_{ij}(f_{jj})^{k-1}. \tag{2}$$
   Hint: Express the event $\{N_j \geq k\}$ in terms of $\tau_j^{(1)}, \ldots, \tau_j^{(k)}$ and note that the initial state is i, not j.
   (b) Assume that j is transient and show that the distribution of N_j with respect to P_j is then geometric, i.e.,
$$P_j[N_j = k] := P[N_j = k | X_0 = i] = (1 - f_{jj})(f_{jj})^k, \tag{3}$$
   (c) Assume that j is recurrent and show that then
$$P_i[N_j = \infty] := P[N_j = \infty | X_0 = i] = f_{ij}, \tag{4}$$
   in particular, $P_j[N_j = \infty] = 1$

10. Assume that the set T of transient states of an MC is finite and set $m := |T|$.
   (a) Argue that T can not be closed. (Hint: use a basic result from class.)
   (b) Conclude that $a := \max_{i \in T} \sum_{k \in T} p_{ik}^{(m+1)}$ is strictly less than one.
   (c) Give a simple rough upper bound for the exit time from T $P[\tau_{T^c} \geq n]$ in terms of a.

11. The Media Police have identified six states associated with television watching: 0 (never watching), 1 (watch occasionally), 2 (watch frequently), 3 (addict), 4 (undergoing behavioral modification), 5 (brain dead). Transitions from state to state can be modelled as an MC with the following transition matrix:
$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ .5 & 0 & .5 & 0 & 0 & 0 \\ .1 & 0 & .5 & .3 & 0 & .1 \\ 0 & 0 & 0 & .7 & .1 & .2 \\ 1/3 & 0 & 0 & 1/3 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{5}$$

(a) Which states are transient, which are recurrent?

(b) Give the canonical decomposition of the state space.

(c) Set $q_i = P[X_n = 5 \text{ for some } n \geq 1 \,|X_0 = i]$. Starting from state 1, we are interested in the chance to enter state 5 before state 0. Show that this chance is exactly equal to q_1.

(d) Using this fact, express this chance in terms of the limiting distribution.

(e) Looking at the rows of P find four relations between $q_1, \ldots, q_4$, e.g., from row 3 we get $q_2 = .5q_2 + .3q_3 + .1$. Solve for q_1.

12. Let S_n denote a simple random walk: $S_n = X_1 + \ldots + X_n$ with X_n i.i.d. and $P[X_n = 1] = 1 - P[X_n = -1] = p = 1 - q$.

    (a) Show that this chain is irreducible, i.e., find for every pair of states i, j an integer n such that $p_{ij}^{(n)} \neq 0$ (the exact value is not needed). Hint: distinguish $i > j$ and $i \leq j$.

    (b) Based on known results on the return to zero from earlier homework decide whether 0 is recurrent or transient. Hint: your answer will dependent on the parameter p.

    (c) *Bonus question* Recall Stirling's formula which implies that

$$\binom{2n}{n} \simeq \frac{4^n}{\sqrt{\pi n}} \tag{6}$$

    as well as the well known relation between geometric and arithmetic means which implies that $pq \leq 1/4$ with equality if and only if $p = q = 1/2$. Now, approximate the probability of passing from zero to zero in $2n$ steps $p_{00}^{(2n)}$ using Stirling's formula and determine, for which values of the parameter p the sum

$$\sum_{k=1}^{\infty} p_{00}^{(k)} \tag{7}$$

    converges. Conclude whether 0 is recurrent or transient depending on p, thus obtaining the same result as before.

13. Let X_n be a Markov Chain.

    (a) Assume that f is a given 1-1 function of the state space, i.e., f is invertible. Show that the sequence of random variables $f(X_n)$ form a Markov Chain as well.

    (b) Show that this is not necessarily true if f is not invertible. Hint: Consider $f(x) = x^2$ and the simple random walk with $p \neq 1/2$.

14. Let $P = [p_{ij}]_{(ij)}$ denote the transition matrix of an MC. Define $q_{ij} = 1$ if $p_{ij} \neq 0$ and $q_{ij} = 0$ else. The matrix $Q = [q_{ij}]_{(ij)}$ indicates whether it is possible to reach j from i in 1 step.

    (a) Show that the matrix Q^2 indicates in how many ways it is possible to reach j from i in 2 steps.

    (b) Assume that S has m states. Explain how the matrix $Q + Q^2 + \ldots + Q^m$ can be used to decide whether j is reachable from i or not.

15. (a) Assume that there exists an integer n such that $p_{ij}^{(n)} \neq 0$ for all $i, j \in S$. Show that the MC is then irreducible.

    (b) *Bonus question* The reverse is not true. Give a simple counter example.