STAT 552 Homework 5

Due date: In class on Tuesday, November 18, 2003

Instructor: Dr. Rudolf Riedi

20. (Extreme values) Let $N(A)$ be a Poisson point process on $\mathbb{R}^+$ with points X_n and with control measure $\mu(A) = \int_A 1/t^2 dt$.

    (a) Show that $P[N((x,\infty]) < \infty] = 1$.

    (b) Conclude that $Y := \max_n(X_n)$ is well defined, i.e., the maximum exists almost surely.

    (c) Show that $P[Y < x] = \exp(-1/x)$ for $x > 0$.

    This is a classical extreme value distribution.

21. (Shot noise) Let w denote a (deterministic) exponential function, i.e., $w(t) = \exp(-\theta t)$ for some given parameter $\theta > 0$. Assume that Γ_n are the points of a homogeneous Poisson point process on $\mathbb{R}^+$ with arrival rate α, i.e., $d\mu(t) = \alpha dt$. Set

$$X(t) = \sum_{i=1}^{N((0,t])} w(t - \Gamma_i).$$

    Such functions w are used to model the electrical current produced by an electron hitting a conductor. The process X gives, thus, the total current at time t.

    (a) Compute the Laplace transform of $X(t)$, i.e., show that

$$\mathbb{E}[\exp(-\lambda X(t))] = \exp\left\{\alpha \int_0^t -(1 - e^{-\lambda w(s)})ds\right\}.$$

    Hint: condition on knowing $N((0,t])$ and exploit the order statistics of the points X_n.

    (b) Compute $\mathbb{E}[X(t)]$.

22. (Cluster processes) Let $(X_n)_{n\geq 1}$ be the points of a Poisson point process $N(A)$ with control measure $\mu(A)$. Let I_n be i.i.d. random variables, non-negative, integer valued and independent of the X_n. Set

$$K(A) := \sum_{n\geq 1} I_n \mathbf{1}_{[X_n \in A]}$$

    The random count measure K is sometimes referred to as a "cluster process". Here, the points X_n model the sites of arrivals while the I_n model the number of arrivals per site. For an example on the real line think of busses arriving to a restaurant at random times in a Poisson fashion, each bringing a random number of customers.

    (a) Show that K is randomly scattered.

    (b) Compute the Laplace transform of $K(A)$ for a fixed A. Hint: condition on knowing $N(A)$.

23. ($M/G/\infty$ queue)

Assume that calls are initiated according to a homogeneous Poisson point process on $(0, \infty)$. Assume call durations are i.i.d. with a common distribution G and independent of call initiation times. Let $A(t)$ denote the number of ongoing calls at time t (initiated before t but not terminated at time t). Show that $A(t)$ has a Poisson distribution for every t. Hint: Fix time t and mark call arrivals according to whether they are clogging a line at time t or not.

24. (Mixed Poisson)

Let N be a homogeneous Poisson process on $[0, \infty)$ with rate α. Let Λ be a positive r.v., independent of N and with distribution G. Denote by $\hat{G}(\lambda) = \int_0^\infty \exp(-\lambda s) dG(s) = \mathbb{E}_G[e^{-\lambda s}]$ the Laplace transform of G.

The process $M(t) := N((0, \Lambda t])$ ($t > 0$ is called *mixed Poisson*. Compute the Laplace transform of $M(t)$. Hint: condition on Λ.

Informal comment: Mixed Poisson processes are the only processes with the order statistics property, i.e., conditioned on knowing the number of points in a set A, the order statistics of the points are the same as the order statistics of i.i.d. uniform random points. Recall that the Poisson point process is the unique point process with the order statistics property and with Poisson distributed number of points.