STAT 552 Homework 3

Due date: In class on Tuesday, September 27 (!), 2005

Instructor: Dr. Rudolf Riedi

8. A Markov chain has state space $S = \{1, 2, \ldots, 8\}$. Starting from $X_0 = 1$, the chain moves in each step from its current state j to any of the larger states $\{k : k > j\}$ with equal probability. State 8 is absorbing.
   (a) Compute the transition matrix.
   (b) Decompose the state space into recurrent and transient classes.
   (c) Find the expected number of steps to reach state 8.

9. Let X_n be a Markov Chain.
   (a) Assume that f is a given 1-1 function of the state space, i.e., f is invertible. Show that the sequence of random variables $f(X_n)$ form a Markov Chain as well.
   (b) Show that this is not necessarily true if f is not invertible. Hint: Consider $f(x) = x^2$ or $f(x) = |x|$ and an MC with only few states.

10. Let $\boldsymbol{P} = [p_{ij}]_{(ij)}$ denote the transition matrix of an MC. Define $q_{ij} = 1$ if $p_{ij} \neq 0$ and $q_{ij} = 0$ else. The matrix $\boldsymbol{Q} = [q_{ij}]_{(ij)}$ indicates whether it is possible to reach j from i in 1 step.
    (a) Show that the matrix $\boldsymbol{Q}^2$ indicates in how many ways it is possible to reach j from i in 2 steps.
    (b) Assume that S has m states. Explain how the matrix $\boldsymbol{Q} + \boldsymbol{Q}^2 + \ldots + \boldsymbol{Q}^m$ can be used to decide whether j is reachable from i or not.

11. (a) Assume that there exists an integer n such that $p_{ij}^{(n)} \neq 0$ for all $i, j \in S$. Show that the MC is then irreducible.
    (b) *Bonus question* The reverse is not true. Give a simple counter example.