ELEC 535 Homework 2

Due date: In class on Friday, January 31, 2003

Instructor: Rudolf Riedi Rice University, Spring 2003

Problem 2.1 (Two looks)

Let X, Y_1 and Y_2 be binary random variables, If $I(X; Y_1) = 0$ and $I(X; Y_2) = 0$, does it follow that $I(X; Y_1, Y_2) = 0$? Prove or provide a counterexample.

Problem 2.2 (Conditional mutual information)

Give examples of joint random variables X, Y and Z such that
(a) $I(X; Y|Z) < I(X; Y)$

(b) $I(X; Y|Z) > I(X; Y)$

Problem 2.3 (Entropy of a sum)

Let X and Y be random variables that take on values $x_1, x_2, \ldots, x_r$ and $y_1, y_2, \ldots, y_s$, respectively. Let $Z = X + Y$.
(a) By direct computation verify that $H(Z|X) = H(Y|X)$.
Using this fact, argue that if X, Y are independent, then $H(Y) \leq H(Z)$.
(Hint: information is always non-negative.)
(b) Give and example in which $H(X) > H(Z)$ and $H(Y) > H(Z)$.
(Hint: try to make $H(Z) = 0$.)
(c) Under what conditions is $H(Z) = H(X) + H(Y)$?

Problem 2.4 (Mixing increases entropy)

Show that the entropy of the probability distribution $(p_1, \ldots, p_i, \ldots, p_j, \ldots, p_m)$, is less than the entropy of the distribution $(q_1, \ldots, q_m)$ where $q_i = q_j = (p_i + p_j)/2$ and $q_k = p_k$ for all $k \neq i, j$. (Hint: compute $H_p - H_q$ from their definition.) Note that there is a general truth to this special result: ”warping” a distribution towards uniformity increases entropy.

Problem 2.5 (Inequalities)

Let X, Y and Z be joint random variables. Using the facts developed in class establish the following inequalities and find conditions for equality.
(a) $H(X, Y|Z) \geq H(X|Z)$.
(b) $I(X, Y; Z) \geq I(X; Z)$.
(c) $H(X, Y, Z) - H(X, Y) \leq H(X, Z) - H(X)$.

Problem 2.6 (Markov's inequality for probabilities)

Let $p(x)$ be a probability mass function. Prove, for all $d \geq 0$,

$$\Pr\{p(X) \leq d\} \log\left(\frac{1}{d}\right) \leq H(X).$$